

Two Quasi 2-Groups

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Section One: Introduction

Recently, I set out to find and classify all the quasi p -groups of order less than 24. For most groups of order less than 24, it was easy to check whether or not the group was a quasi p -group for some prime p . But there were two of these groups, one of order 18 and one of order 20, that required much more thought and analysis. The analysis of these two groups explores the interesting structure of the semidirect product. For a brief introduction to the semidirect product, see [AbC].

The purpose of this paper is to examine these two groups and show that they are both quasi 2-groups. The remaining groups of order less than order 24 are described in [Hwd].

Section Two: So what exactly is a quasi p -group?

It doesn't do us much good to talk about quasi p -groups if we don't know what quasi p -groups are. So we begin with the definition.

Definition (1.1) Let G be a finite group. We say G is a quasi p -group if any of the following are true:

- 1) $G = p(G)$, where $p(G)$ is the (normal) subgroup of G that is generated by the p -Sylow subgroups of G .
- 2) G is generated by all of its elements whose orders are powers of p .
- 3) G has no nontrivial quotient group whose order is prime to p .

All three of these definitions are equivalent. For a proof of this equivalence, see [Hwd].

So, having defined a quasi p -group, let us now examine the first of our two groups.

Section Three: $Z_5 \times_{\varphi} Z_4$

Let $G \cong Z_5 \times_{\varphi} Z_4$, where in this case the homomorphism $\varphi : Z_4 \rightarrow \text{Aut}(Z_5)$ maps elements to inverses: $\varphi(x)(y) = y^{-1}$ where $x \in Z_4$ and $y \in Z_5$.

Proposition (3.1): $G \cong Z_5 \times_{\varphi} Z_4$ is a quasi 2-group

Proof: This amounts to showing that G is generated by elements whose orders are a power of 2.

The order of G is 20. Five of these 20 elements are in the Z_5 factor. This leaves 15 elements. We first show that there are 5 2-Sylow subgroups, each of which has order 4, and that these subgroups generate G .

The Third Sylow Theorem tells us that n_2 , the number of 2-Sylow subgroups of G , is congruent to 1 mod 2 ; i.e. that n_2 is an odd number. So our options for n_2 are

$$1, 3, 5, 7, 9, 11, 13, 15, 17, \text{ or } 19.$$

But, also by the Third Sylow Theorem, $n_2 \mid 20$. So the possibilities for n_2 now become

$$1 \text{ or } 5.$$

Because Z_4 is not normal in G , $n_2 = 5$. So, the remaining 15 elements of G are the fifteen nonidentity elements of the 5 2-Sylow subgroups.

Now, let $Z_5 = \langle y \rangle = \{1, y, y^2, y^3, y^4\} = H$ and $Z_4 = \langle x \rangle = \{1, x, x^2, x^3\}$. Consider the following cosets of H :

$$H = \{1, y, y^2, y^3, y^4\}$$

$$xH = \{x, xy, xy^2, xy^3, xy^4\}$$

$$x^2H = \{x^2, x^2y, x^2y^2, x^2y^3, x^2y^4\}$$

$$x^3H = \{x^3, x^3y, x^3y^2, x^3y^3, x^3y^4\}.$$

The 15 elements of $xH \cup x^2H \cup x^3H$ are the 15 nonidentity elements of the union of the 2-Sylow subgroups of G . In particular, each of them has order a power of 2. It is known that the action of Z_4 on Z_5 becomes conjugation in the semidirect product, i.e.

$xyx^{-1} = y^{-1}$. Notice that,

$$(xy)(x^3y^2) = (xyx^3)y^2 = (xyx^{-1})y^2 = y^{-1}y^2 = y,$$

and y is the generator for H . So the elements of G whose order is a power of 2 not only generate the 2-Sylow subgroups, but H as well. Therefore they generate G . So G is a quasi 2-group.

Section Four: $(Z_3 \times Z_3) \rtimes_{\varphi} Z_2$

Now let $G \cong (Z_3 \times Z_3) \rtimes_{\varphi} Z_2$, where the homomorphism $\varphi: Z_2 \rightarrow \text{Aut}(Z_3 \times Z_3)$

maps elements to inverses, and our operations are addition.

Proposition (4.1): $G \cong (Z_3 \times Z_3) \rtimes_{\varphi} Z_2$ is a quasi 2-group.

Proof: As before we must show that G is generated by its elements whose orders are powers of 2.

Let the elements of G be written as follows:

$$g = [(x, y), z].$$

The order of G is 18. Nine of the 18 elements are accounted for in the $(Z_3 \times Z_3)$ factor, which leaves 9. As before, we show that there are 9 2-Sylow subgroups of G and that these subgroups generate G .

First, let us consider the orders of the elements of G .

By definition of the semidirect product, we have that for

$$g_1 = [(x_1, y_1), z_1], g_2 = [(x_2, y_2), z_2] \in G,$$

$$g_1 + g_2 = [(x_1, y_1), z_1] + [(x_2, y_2), z_2] = [(x_1, y_1) + \varphi(z_1)(x_2, y_2), z_1 + z_2].$$

Obviously, the order of $[(0,0), 0]$ is 1.

We now consider the case where $[(x, y), 0]$, with $(x, y) \neq (0,0)$. Notice that

$$\varphi(0)(x, y) = (x, y) \text{ and } \varphi(1)(x, y) = (-x, -y). \text{ Then we have}$$

$$[(x, y), 0] \neq [(0,0), 0]$$

$$2[(x, y), 0] = [(x, y), 0] + [(x, y), 0] = [((x, y) + \varphi(0)(x, y)), 0 + 0]$$

$$= [(x, y) + (x, y), 0] = [(2x, 2y), 0] \neq [(0,0), 0]$$

$$3[(x, y), 0] = 2[(x, y), 0] + [(x, y), 0] = [(2x, 2y), 0] + [(x, y), 0]$$

$$= [((2x, 2y) + \varphi(0)(x, y)), 0 + 0] = [(2x, 2y) + (x, y), 0] = [(3x, 3y), 0] = [(0,0), 0]$$

So the order of $[(x, y), 0]$ with $(x, y) \neq (0,0)$ is 3.

Next we consider the element $[(0,0), 1]$.

$$[(0,0), 1] \neq [(0,0), 0]$$

$$2[(0,0), 1] = [(0,0), 1] + [(0,0), 1] = [(0,0) + \varphi(1)(0,0), 1 + 1] = [(0,0) + (0,0), 0] = [(0,0), 0]$$

So $[(0,0), 1]$ has order 2.

Finally, consider the element $[(x, y), 1]$ with $(x, y) \neq (0,0)$.

$$[(x, y), 1]$$

$$2[(x, y), 1] = [(x, y), 1] + [(x, y), 1] = [(x, y) + \varphi(1)(x, y), 1 + 1]$$

$$= [(x, y) + (-x, -y), 0] = [(0,0), 0].$$

So $[(x, y), 1]$ with $(x, y) \neq (0, 0)$ has order 2 as well. Therefore, each of the elements of G has order 1, 2, or 3. There is one element of order 1: $[(0, 0), 0]$; 8 elements of order 3: $[(1, 0), 0]$, $[(2, 0), 0]$, $[(1, 1), 0]$, $[(2, 1), 0]$, $[(1, 2), 0]$, $[(2, 2), 0]$, $[(0, 1), 0]$, and $[(0, 2), 0]$. The remaining 9 elements each have order 2. These 9 elements generate G .

Notice that $[(2, 0), 1]$ and $[(1, 0), 1]$ are elements of order 2 and

$$[(2, 0), 1] + [(1, 0), 1] = [((2, 0) + (-1, 0)), 1 + 1] = [(1, 0), 0].$$

Similarly, $[(0, 2), 1]$ and $[(0, 1), 1]$ are elements of order 2 and

$$[(0, 2), 1] + [(0, 1), 1] = [((0, 2) + (0, -1)), 1 + 1] = [(0, 1), 0].$$

Now it is easy to see that G is a quasi 2-group.

Section Five: Acknowledgements

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References

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