

APPLICATION OF GENERALIZED INVERSES AND CIRCULANT MATRICES TO ITERATED FUNCTIONS ON INTEGERS

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ABSTRACT. After reviewing fundamental properties of matrix generalized inverses and circulant matrices, we describe a particular application in which both of these concepts play an important role. In particular, we establish the form of the Moore-Penrose generalized inverse of a type of circulant matrix that arises naturally in the study of iterated functions on integers.

1. INTRODUCTION

In this paper we will provide a brief introduction to matrix generalized inverses and circulant matrices. We will then describe how these concepts pertain to the study of iterated functions on the positive integers.

A generalized inverse of a singular matrix A is a matrix X which has some of the same properties that hold for the inverse of a nonsingular matrix. In section 2, we present an overview of two special types of generalized inverses, which have been applied to the study of singular systems of nonlinear differential equations, a variety of problems in numerical analysis, and the modeling of electrical networks [1]. In section 3, we define circulant matrices and describe some of their basic properties. The concepts of generalized inverses and circulant matrices are then brought together in section 4 where we outline their application to the study of iterated functions on integers. Section 5 is devoted to the proof of a result that incorporates all of these concepts.

2. GENERALIZED INVERSES

Generalized inverses are matrices satisfying some of the characteristic properties of inverses. They may be thought of as pseudoinverses for non-invertible matrices. In fact, the Moore-Penrose Inverse, denoted by A^\dagger (read “ A dagger”), is a special type of generalized inverse which is sometimes referred to as *the* pseudoinverse of A . We introduce this pseudoinverse by first reviewing the idea of a singular value decomposition. In what follows, the expression $A \in \mathbb{C}^{(m \times n)}$ (or $\mathbb{R}^{(m \times n)}$) means that A is an $m \times n$ matrix with complex (or real) entries, and $*$ denotes the conjugate transpose. Also recall that for $A \in \mathbb{C}^{(m \times n)}$, A^*A has real nonnegative eigenvalues.

Proposition 1. (*Singular Value Decomposition*) Let $A \in \mathbb{C}^{(m \times n)}$ and let D be an $m \times n$ matrix with entries $(D)_{ij} = 0$ for $i \neq j$ and whose diagonal entries are the nonnegative square roots of the eigenvalues of A^*A , repeated according to

multiplicity. Then there exist unitary matrices $U \in \mathbb{C}^{(m \times m)}$ and $V \in \mathbb{C}^{(n \times n)}$ such that $A = UDV^*$.

The idea of a singular value decomposition may be clarified by an example¹.

Example 1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & i \\ 1 & 1 \end{bmatrix}$. Then $A^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -i & 1 \end{bmatrix}$ and $A^*A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. The

eigenvalues of A^*A are 3 and 1, and $A = UDV^*$ where $U = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -\sqrt{3} & -\sqrt{2} \\ i & i\sqrt{3} & -i\sqrt{2} \\ 2 & 0 & \sqrt{2} \end{bmatrix}$,

$D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

The Moore-Penrose (generalized) inverse may now be defined in terms of a singular value decomposition.

Definition 1. Let $A \in \mathbb{C}^{(m \times n)}$ and let $A = UDV^*$ be a singular value decomposition for A . Define D^\dagger (read “ D dagger”) to be the $n \times m$ matrix with entries $(D^\dagger)_{ii} = \frac{1}{(D)_{ii}}$ if $(D)_{ii} \neq 0$ and $(D^\dagger)_{ij} = 0$ otherwise. Then the Moore-Penrose inverse of A , denoted A^\dagger , is the $n \times m$ matrix defined by

$$A^\dagger = VD^\dagger U^*$$

Note that this definition implies that every matrix in $\mathbb{C}^{(m \times n)}$ has a Moore-Penrose inverse. But, even though a matrix A may have more than one singular value decomposition, the Moore-Penrose inverse is unique.

Example 2. From our definition of the Moore-Penrose inverse, if A is the matrix of Example 1 then

$$A^\dagger = \frac{1}{3} \begin{bmatrix} 2 & i & 1 \\ -1 & -2i & 1 \end{bmatrix}$$

The Moore-Penrose inverse may also be characterized as the unique $n \times m$ matrix satisfying the following algebraic properties. Since this was the definition introduced by R. Penrose, these have come to be known as the Penrose equations.

$$AA^\dagger A = A \tag{1a}$$

$$A^\dagger AA^\dagger = A^\dagger \tag{1b}$$

$$(AA^\dagger)^* = AA^\dagger \tag{1c}$$

$$(A^\dagger A)^* = A^\dagger A, \tag{1d}$$

Uniqueness of A^\dagger follows from these conditions, as does the important property that the pseudoinverse of the pseudoinverse of a matrix A is A ($(A^\dagger)^\dagger = A$). Additional details on singular value decomposition and the Moore-Penrose inverse may be found in [1, sec.6.2], or in any of a number of texts on linear algebra.

We now restrict our attention to square, real-valued matrices, and introduce another generalized inverse that is important in a variety of applications.

¹The singular value decompositions and Moore-Penrose inverses presented in this document were obtained through use of MATLAB.

If $A \in \mathbb{R}^{(n \times n)}$ commutes with A^\dagger , then the Moore-Penrose inverse is called the group inverse of A , denoted $A^\#$ (read “ A sharp”). More generally, the group inverse, when it exists, is characterized as the unique matrix that commutes with A and satisfies the first two Penrose equations.

Definition 2. For an $n \times n$ matrix A , $A^\#$ is the group inverse of A if it satisfies the following conditions.

$$A^\# A = A A^\# \tag{2a}$$

$$A A^\# A = A \tag{2b}$$

$$A^\# A A^\# = A^\# \tag{2c}$$

The name “group inverse” derives from the fact that the set of all powers of A and $A^\#$ together with $A A^\#$ as the identity element, form an abelian group [1]. Let’s look at few examples.

Example 3. If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ then $A^\dagger = A^\# = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Example 4. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $A^\dagger = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, but A is idempotent so $A^\# = A$.

Example 5. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $A^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, but A has no group inverse.

As these examples illustrate, it is possible that $A^\#$ either exists and is equal to A^\dagger , exists and is different from A^\dagger , or does not exist at all. In fact, $A^\#$ exists if and only if $\text{rank}(A) = \text{rank}(A^2)$, and in this case, the defining equations then imply that $A^\#$ is unique. Moreover, $A^\# = A^\dagger$ if and only if $\text{range}(A) = \text{range}(A^T)$ [1, pp.162-163].

3. CIRCULANT MATRICES

In order to discuss circulant matrices, it is convenient to label the entries of an $n \times n$ matrix with double subscripts ranging from 0 to $n - 1$, instead of the usual 1 to n . We will adopt this convention and, for reasons that will become clear below, we also require throughout that matrix indices be interpreted $(\text{mod } n)$.

Definition 3. An $n \times n$ matrix A is circulant if and only if $(A)_{ij} = (A)_{0,j-i}$ for every i, j .

Note that it is equivalent to say that a matrix A is circulant when $(A)_{ij} = (A)_{i+1,j+1}$. This means that a matrix is circulant if and only if its second row is identical to its first row, but pushed over one space to the right, its third row is identical to its second row, but pushed over by one space to the right, etc. In other words, $a_{00} = a_{11}$, $a_{01} = a_{12}$, etc. Thus, a circulant matrix is completely determined by the entries in the first row. For a circulant matrix A with entries in the first row given by a_0, a_1, \dots, a_{n-1} , we write $A = \text{circ}(a_0, a_1, \dots, a_{n-1})$. For example, the matrix $Q = \text{circ}(1, 2, 3)$ is :

$$Q = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

It turns out that a circulant times a circulant is a circulant, i.e., the set of $n \times n$ circulants is closed under multiplication. For example, consider the case $n = 3$. We'll multiply $A = \text{circ}(a_0, a_1, a_2)$ with $B = \text{circ}(b_0, b_1, b_2)$ to get $C = \text{circ}(c_0, c_1, c_2)$. In this case we discover that:

$$\begin{bmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{bmatrix} = \begin{bmatrix} a_0b_0 + a_1b_2 + a_2b_1 & a_0b_1 + a_1b_0 + a_2b_2 & a_0b_2 + a_1b_1 + a_2b_0 \\ a_2b_0 + a_0b_2 + a_1b_1 & a_2b_1 + a_0b_0 + a_1b_2 & a_2b_2 + a_0b_1 + a_1b_0 \\ a_1b_0 + a_2b_2 + a_0b_1 & a_1b_1 + a_2b_0 + a_0b_2 & a_1b_2 + a_2b_1 + a_0b_0 \end{bmatrix}$$

It is apparent that the resulting product is circulant. This holds in general, as we will now prove.

Proposition 2. *Let A and B be $n \times n$ circulant matrices. Then the product AB is also a circulant matrix.*

Proof. Let $A = \text{circ}(a_0, a_1, \dots, a_{n-1})$, $B = \text{circ}(b_0, b_1, \dots, b_{n-1})$, and $C = AB$. To demonstrate that C is a circulant, we need to show that $c_{ij} = c_{(i+1)(j+1)}$. So, let's consider the element c_{ij} . We know from matrix multiplication that

$$c_{ij} = \sum_{k=0}^{n-1} a_{ik}b_{kj}$$

and

$$c_{(i+1)(j+1)} = \sum_{k=0}^{n-1} a_{(i+1)k}b_{k(j+1)}.$$

By shifting the index of summation, we get

$$c_{(i+1)(j+1)} = \sum_{k=-1}^{n-2} a_{(i+1)(k+1)}b_{(k+1)(j+1)}$$

Since $a_{(i+1)(k+1)} = a_{ik}$ and $b_{(k+1)(j+1)} = b_{kj}$, we find that $c_{(i+1)(j+1)} = c_{ij}$, so C is circulant. \square

Circulant matrices also commute under multiplication.

Proposition 3. *Let A and B be $n \times n$ circulant matrices. Then $AB = BA$.*

Proof. We know from Proposition 1 that $C = AB$ is circulant, so we'll denote C by $\text{circ}(c_0, c_1, \dots, c_{n-1})$. Then, for $j = 0 \dots (n-1)$,

$$\begin{aligned} c_j &= (AB)_{0j} \\ &= \sum_{k=0}^{n-1} (A)_{0k}(B)_{kj} \\ &= \sum_{k=0}^{n-1} a_k b_{j-k} \\ &= \sum_{k=0}^{n-1} b_{j-k} a_k \end{aligned}$$

The final summation may be recognized as element j of the convolution of the two sequences $\{b_0, b_1, \dots, b_{n-1}\}$ and $\{a_0, a_1, \dots, a_{n-1}\}$. The result then follows from the well-known fact that the convolution is commutative. \square

The convolution expression for the elements of AB provides a convenient shorthand method of multiplying circulant matrices, described by Davis [4]. If $A = \text{circ}(a_0, a_1, \dots, a_{n-1})$ and $B = \text{circ}(b_0, b_1, \dots, b_{n-1})$, then we tabulate all possible products $a_i b_j$ as follows:

$$\begin{array}{c} a_0, a_1, \dots, a_{n-1} \\ b_0, b_1, \dots, b_{n-1} \\ \hline a_0 b_0, a_1 b_0, \dots, a_{n-1} b_0 \\ a_{n-1} b_1, a_0 b_1, \dots, a_{n-2} b_1 \\ \vdots \\ a_1 b_{n-1}, a_2 b_{n-1}, \dots, a_0 b_{n-1} \end{array}$$

We then sum along the resulting columns to obtain the entries of the product, so, $AB = \text{circ}(a_0 b_0 + \dots + a_1 b_{n-1}, a_1 b_0 + \dots + a_2 b_{n-1}, \dots, a_{n-1} b_0 + \dots + a_0 b_{n-1})$.

4. APPLICATION TO ITERATED FUNCTION SYSTEMS

In this section, we describe the application of generalized inverses and circulant matrices to the study of iterated functions on the integers.

Generalized inverses are of great importance in the study of finite Markov chains, which are discrete stochastic processes in which the probability of transit from one state to another is determined only by the current state [5]. A Markov chain is characterized by its transition matrix T , where T_{ij} represents the probability of transiting from state i to state j . If $A = I - T$, then A has a group inverse $A^\#$ and $A^\#$ contains much of the relevant information about the Markov chain [3,6]. In particular $\lim_{k \rightarrow \infty} T^k = I - AA^\#$ whenever the limit exists. Knowledge of $A^\#$ therefore provides information about the long-term behavior of the Markov chain. It is important to note that in this application $A^\#$ is not generally the same as A^\dagger .

Now consider a function f defined on the set $\mathbb{N} = \{1, 2, 3, \dots\}$. For a given integer n , we define the *trajectory* of n under f to be the set of iterates $\{f^{(k)}(n)\}_{k=0}^\infty$. The collection of all trajectories defines an iterated function system (IFS) on the set \mathbb{N} . It is natural to then search for a means to describe the essential behavior of the IFS. Since the probability of transiting from i to j is 1 if $f(i) = j$ and 0 otherwise, an IFS may be viewed as an infinite Markov chain, with all probabilities either 0 or 1. There is also an infinite version of the group inverse $A^\#$, which is related to the corresponding transition matrix as in the finite case. Zarnowski [7] described this application of the group inverse to the analysis of IFSs on $\mathbb{N} = \{1, 2, 3, \dots\}$ in which all trajectories are either divergent or eventually fixed. Complications arise in the algebra of infinite matrices, but the special case of probabilities either 0 or 1 makes it possible to extract more information from $A^\#$ than can be done for a general Markov chain.

One of the difficulties in this application arises from the need to describe the effect on $A^\#$ of the presence of cycles in the IFS. This motivates our study in the next section of the group inverse $(I - T)^\#$, where T is a transition matrix representing a cycle.

5. WHAT ABOUT CYCLES?

It is interesting to analyze the consequences of allowing cycles within a transition matrix T , because of the connection with IFSs described in the previous section.

Although we can't address the complexities that arise with infinite matrices here, we can examine what happens with isolated finite cycles. As we will show, if T is a transition matrix corresponding to a cycle and $A = I - T$ then the group inverse $A^\#$ is identical to the Moore-Penrose inverse A^\dagger . In the examples that follow, these generalized inverses could therefore be computed using MATLAB, which has an algorithm for computing A^\dagger . However, since this study was motivated by the more general IFS application in which $A^\#$ is the important quantity (and is not the same as A^\dagger), we will retain the use of $A^\#$ in what follows.

As an example, what if T somehow involved the cycle (4, 2) and the cycle (1, 3, 5) and took on the form:

$$T_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

or, what if it allowed the cycle (5, 2, 3, 1, 4) and were expressed as:

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

What would $A^\#$ look like then? Before we begin investigating resulting forms of the group inverse, we notice these transition matrices appear a bit cluttered. By a permutation of states, it is sufficient to assume only ordered cycles. That is, instead of considering the two cycles (4, 2) and (1, 3, 5) in T_1 , we use the two cycles (1, 2) and (3, 4, 5). Similarly for the transition matrix T_2 , we may instead use (1, 2, 3, 4, 5). One can easily verify from the defining properties of $A^\#$ that the group inverse of $A_1 = I - T_1$ (where we are using the new, ordered T_1) is

$$A_1^\# = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

and the group inverse of $A_2 = I - T_2$ (where we again use the ordered version of T_2) is

$$A_2^\# = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 & -\frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

We notice a pattern emerging, and upon closer inspection we find that in general, it appears that for a $k \times k$ transition matrix corresponding to the k -cycle (1, 2, ..., k),

$$A_{ij}^\# = \frac{k-1}{2k} - \frac{1}{k}(j-i) \bmod(k)$$

where $i, j < k$. Using previously established properties of circulants we can prove this holds in general. Notice that in the example above, $A_1^\#$ is a block diagonal

matrix, and each block is circulant. Because of this, we may independently examine the particular circulant matrix associated with a particular cycle. The following theorem is based on this premise.

Theorem 1. *For the $k \times k$ transition matrix T corresponding to the k -cycle $(1, 2, \dots, k)$, the elements of the group inverse of $A = I - T$ are given by*

$$A_{ij}^{\#} = \frac{k-1}{2k} - \frac{1}{k}(j-i) \bmod(k)$$

where $0 \leq i, j < k$.

Thus, this particular type of group inverse forms a circulant matrix:

$$A^{\#} = \text{circ}\left(\frac{k-1}{2k}, \frac{k-1}{2k} - \frac{1}{k}, \dots, \frac{k-1}{2k} - \frac{k-1}{k}\right)$$

To actually verify that this form of $A^{\#}$ is a group inverse, we only need to verify conditions (2b) and (2c), since (2a) will follow from Proposition 2.

Proof. We know that A has the following form:

$$A = \text{circ}(1, -1, 0, \dots, 0)$$

So, first let's calculate $AA^{\#}$, using the shorthand method described earlier:

$$\begin{array}{c} 1, -1, 0, \dots, 0 \\ \frac{k-1}{2k}, \frac{k-1}{2k} - \frac{1}{k}, \frac{k-1}{2k} - \frac{2}{k}, \dots, \frac{k-1}{2k} - \frac{k-1}{k} \\ \frac{k-1}{2k}, -\frac{k-1}{2k}, 0, \dots, 0 \\ 0, \frac{k-1}{2k} - \frac{1}{k}, -\frac{k-1}{2k} + \frac{1}{k}, \dots, 0 \\ 0, 0, \frac{k-1}{2k} - \frac{2}{k}, \dots, 0 \\ \vdots \\ -\frac{k-1}{2k} + \frac{k-1}{k}, 0, 0, \dots, \frac{k-1}{2k} - \frac{k-1}{k} \\ \frac{k-1}{k}, -\frac{1}{k}, -\frac{1}{k}, \dots, -\frac{1}{k} \end{array}$$

So, $AA^{\#} = \text{circ}\left(\frac{k-1}{k}, -\frac{1}{k}, -\frac{1}{k}, \dots, -\frac{1}{k}\right)$. Next we calculate $AA^{\#}A$ by multiplying our previous result by A .

$$\begin{array}{c} \frac{k-1}{k}, -\frac{1}{k}, -\frac{1}{k}, \dots, -\frac{1}{k} \\ 1, -1, 0, \dots, 0 \\ \frac{k-1}{k}, -\frac{1}{k}, -\frac{1}{k}, \dots, -\frac{1}{k} \\ \frac{1}{k}, -\frac{k-1}{k}, \frac{1}{k}, \dots, \frac{1}{k} \\ 0, 0, 0, \dots, 0 \\ \vdots \\ 0, 0, 0, \dots, 0 \\ 1, -1, 0, \dots, 0 \end{array}$$

Thus, $AA^{\#}A = \text{circ}(1, -1, 0, \dots, 0) = A$. Now, we need to show that $A^{\#}AA^{\#} = A^{\#}$. But, since circulant matrices commute, we're already halfway there! We simply need to multiply $AA^{\#}$ by $A^{\#}$.

$$\begin{array}{c} \frac{k-1}{k}, -\frac{1}{k}, -\frac{1}{k}, \dots, -\frac{1}{k} \\ \frac{k-1}{2k}, \frac{k-1}{2k} - \frac{1}{k}, \frac{k-1}{2k} - \frac{2}{k}, \dots, \frac{k-1}{2k} - \frac{k-1}{k} \\ -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{1}{k}\right), \frac{k-1}{k}\left(\frac{k-1}{2k} - \frac{1}{k}\right), -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{1}{k}\right), \dots, -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{1}{k}\right) \end{array}$$

$$\begin{array}{c}
-\frac{1}{k}\left(\frac{k-1}{2k} - \frac{2}{k}\right), -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{2}{k}\right), \frac{k-1}{k}\left(\frac{k-1}{2k} - \frac{2}{k}\right), \dots, -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{2}{k}\right) \\
\vdots \\
-\frac{1}{k}\left(\frac{k-1}{2k} - \frac{k-1}{k}\right), -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{k-1}{k}\right), -\frac{1}{k}\left(\frac{k-1}{2k} - \frac{k-1}{k}\right), \dots, \frac{k-1}{k}\left(\frac{k-1}{2k} - \frac{k-1}{k}\right) \\
\hline
\frac{k-1}{2k}, \frac{k-1}{2k} - \frac{1}{k}, \frac{k-1}{2k} - \frac{2}{k}, \dots, \frac{k-1}{2k} - \frac{k-1}{k}
\end{array}$$

This means that $A^\#AA^\# = \text{circ}\left(\frac{k-1}{2k}, \frac{k-1}{2k} - \frac{1}{k}, \frac{k-1}{2k} - \frac{2}{k}, \dots, \frac{k-1}{2k} - \frac{k-1}{k}\right) = A^\#$ and $A^\#$ is indeed a group inverse! \square

Furthermore, we notice from the form of $AA^\# = \text{circ}\left(\frac{k-1}{k}, -\frac{1}{k}, -\frac{1}{k}, \dots, -\frac{1}{k}\right)$, that $AA^\#$ is real and symmetric; thus, all four Penrose conditions (1a)-(1d) are satisfied, and, in this case, the group inverse is additionally the Moore-Penrose inverse.

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