Topspin: Solvability of sliding number games

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Abstract: The puzzle *Topspin* is a sliding number game consisting of an oval track containing a random arrangement of numbered discs, and a small turnstile within the track. A game is of the form $[t, n]$ if it has n total discs, and a turnstile with t discs. Using concepts of group theory, the solvability, or ordering, of the discs is determined or conjectured for all values of t and n. Furthermore, if a game is not solvable, its attainable subgroup is determined or conjectured for all values of t and n. Several notations are used in the proofs of these theorems to help the reader follow visually as well as mathematically. Solvability is difficult to prove, but in the puzzle $[t, n]$ where t and n are both even, we reveal the complex series of flips and shifts needed to prove the solvability of the game. Finally, using the results of the $[t, n]$ games, the solvability is determined or conjectured for multiple turnstile games.

1 Introduction

The puzzle *Topspin* is a sliding number game typically played by little kids on long car trips. The game usually consists of an oval track of twenty numbered discs and a small turnstile within the track which contains four of the discs. Through a series of shifting and flipping moves, the object of the game is to transform a random arrangement of the discs into an ordered arrangement.

The generalized Topspin puzzle of size $[t, n]$ consists of an oval track of n discs and a turnstile holding $2 \le t < n$ discs. We will call the *start position* of a game the clockwise ordered arrangement of the discs where the disc numbered 1 is in the leftmost slot of the turnstile. For example, the start postion of the $[4, 20]$ game is

	19	20	**1**	**2**	**3**	**4**	5	6	
18									7
17									8 .
	16	15	14	13	12	11	10	9	

The bold face “discs” are in the turnstile.

An arrangement of discs can be considered an element of the symmetric group S_n. For example, the arrangement

	3	5	**9**	**20**	**13**	**17**	4	10	
6									8
12									18
	19	14	16	7	11	2	1	15	

corresponds to the permutation σ which satisfies $\sigma(1) = 9$, $\sigma(2) = 20$, $\sigma(3) = 13$, $\cdots$, $\sigma(20) = 5$. In cycle notation,

$$\sigma = (1, 9, 15, 14, 16, 19, 3, 13, 7, 8, 18, 6, 10)(2, 20, 5, 4, 17, 12, 11).$$

A game is called *solvable* if every permutation in S_n corresponds to a position attainable from the start position through a series of shifting and flipping moves. If a game is not solvable then its *attainable subgroup* is the subgroup of S_n of permutations that are attainable by playing the game.

We know the solvability of many of the games, and have conjectured results for the rest. Similarly, we can find the attainable subgroups for many of the unsolvable games, and have conjectured results for most of the remainaing games.

Theorem 1.1 *Let $2 \leq t < n$. Then in each of the following situations the game of size $[t, n]$ is not solvable:*

1. *n is odd and $t \equiv 1$ mod 4;*
2. *n is odd and $t \equiv 0$ mod 4;*
3. *n is even and t is odd;*
4. *$n \geq 4$ and $t = n - 1$.*

Theorem 1.2 *The game of size $[t, n]$ is solvable if either $n \geq 3$ and $t = 2$, or n and t are both even.*

Conjecture 1.3 *Let $2 \leq t < n$. Then in each of the following situations, the game of size $[t, n]$ is solvable:*

1. *n is odd and $t \equiv 3$ mod 4;*
2. *n is odd, $t \leq n - 2$, and $t \equiv 2$ mod 4.*

Theorem 1.4 *If $n \geq 4$ and $t = n-1$ then the attainable subgroup for the game $[n - 1, n]$ is D_{2n}, the dihedral group of order $2n$.*

Conjecture 1.5 *The attainable subgroup for the game $[t, n]$ is A_n, the alternating subgroup of S_n, when n is odd and either $t \equiv 1$ mod 4 or $t \equiv 0$ mod 4 and $t \neq n - 1$ (when $t = 4$ the attainable subgroup has been proved to be A_n [1]).*

Note that while we can prove that if n is even, t is odd, and $t \neq n - 1$ we do not even have a conjecture about the attainable subgroup for the game $[t, n]$.

This research was inspired by John Wilson who submitted the idea to the *College Mathematics Journal* as a student research project [2]. We began our research as a project for an Abstract Algebra course at St. Olaf College, and continued it as a senior research project under the direction of Professor Jill Dietz.

We thank the referee for a careful reading of this paper and for some useful suggestions.

During preparation of this paper, an article appeared in *Math Horizons* by Curtis Bennett [1]. Some of our results duplicate his, but our paper considers more general games.

The paper is organized as follows: in Section 2 we introduce some notation that will be used to prove the theorems; in Section 3 we address the solvability of games; in Section 4 we discuss attainable subgroups; in Section 5 we supply evidence which supports our conjectures; in Section 6 we conclude with a description of alternative games and some results about them.

2 Notation

There are three basic moves one can make while playing Topspin: the *shift left*, which moves each disc one position clockwise; the *shift right*, which moves each disc one position counterclockwise; and the *flip*, which moves the turnstile by 180 degrees and reverses the order of the t discs sitting in the turnstile.

The shift left is represented by the permutation

$$\alpha = (1, 2, 3, \cdots, n-1, n) \in S_n,$$

and the shift right is represented by $\beta = \alpha^{-1}$. The flip is represented by a product of transpositions as follows:

$$\delta = (1, t)(2, t-1)(3, t-2) \cdots (\lfloor \frac{t}{2} \rfloor, \lfloor \frac{t}{2} \rfloor + 1).$$

Every cycle in S_n can be written as a product of disjoint transpositions. There can be many such products for a given cycle, but the parity of the number of transpositions involved is invariant. The parity of α is odd if and only if n is even. The parity of δ is odd if and only if t is congruent to either 2 or 3 mod 4.

Note that α, β and δ represent *moves* rather than the numbers *on* the discs. For example, if the discs in a $[4, 20]$ game are in position

	3	5	**9**	**20**	**13**	**17**	4	10	
6									8
12									18
	19	14	16	7	11	2	1	15	

then an application of β will move the discs to

	6	3	**5**	**9**	**20**	**13**	17	4	
12									10
19									8
	14	16	7	11	2	1	15	18	

.

An application of δ results in the configuration

	3	5	**17**	**13**	**20**	**9**	4	10	
6									8
12									18
	19	14	16	7	11	2	1	15	

.

We will need to differentiate between a *move* and the actual *positions* of the discs. Let L represent the position of the discs after the permutation α has been applied. R (and F) will represent the discs after β (and δ) have been applied. For example, in the game $[4, 20]$, $(FL)^0$ will be the start position

$$\begin{array}{cccccccccc} & 19 & 20 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & 5 & 6 & \\ 18 & & & & & & & & & 7 \\ 17 & & & & & & & & & 8 \\ & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & \end{array},$$

while $(FL)^1$ will be the position

$$\begin{array}{cccccccccc} & 20 & 4 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{5} & 6 & 7 & \\ 19 & & & & & & & & & 8 \\ 18 & & & & & & & & & 9 \\ & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & \end{array}.$$

Note that we read products left-to-right, so we apply F first then L in the computation $(FL)^1$. For ease of notation, we will also denote the positions above as

$$\begin{array}{rcl} (FL)^0 & = & \cdots, 19, 20, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5, 6, \cdots \\ (FL)^1 & = & \cdots, 20, 4, \mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{5}, 6, 7, \cdots \end{array}$$

Every possible move is a combination of α's, β's, and δ's. Since $\beta = \alpha^{-1}$, it is easy to see that the attainable subgroup for a given game is $S(t, n) = \langle \alpha, \delta \rangle$, the group generated by α and δ. From basic symmetric group theory we know that a game is solvable if and only if it is possible to transpose any two adjacent discs while leaving the others fixed.

3 Solvability

In this section we will prove Theorems 1.1 and 1.2

We begin by looking at the case where both t and n are odd. When $t \equiv 1 \mod 4$, δ has even parity as does α. Hence any product of $\alpha's$ and $\delta's$ will also have even parity and $S(t, n) \neq S_n$. Thus, case 1 of Theorem 1.1 is proved.

Now suppose t is even and n is odd. If $t \equiv 0 \mod 4$ then both α and δ have even parity. As in case 1a, $S(t, n)$ will consist only of permutations of even parity so will not be all of S_n, and case 2 of Theorem 1.1 is proved.

Now suppose t is odd and n is even. If we begin in the start position, odd numbered discs and even numbered discs alternate. Shifting discs left or right will not change the alternating nature of the discs. Since the turnstile contains an odd number of discs, flipping will also not change the alternating pattern. Thus, we will never get two odds or two evens adjacent to one another and the game is not solvable. Case 3 of Theorem 1.1 is proved.

Consider case 4 of Theorem 1.1. When $6 \leq t = n - 1$, Claim 4.1 below will show that the game is not solvable and will further compute the attainable subgroup.

If both t and n are even then we will show that $S(t,n) = S_n$.

Claim 3.1 *Applying the permutation* $\gamma = (\delta\alpha)^{n-t+1}\alpha(\delta\beta\delta\alpha)^{\frac{t-2}{2}}\delta\beta$ *to the start position will result in the transposition of adjacent discs* t *and* $t+1$.

PROOF:

$$\begin{aligned}
(FL)^0 &= \cdots, n-1, n, \mathbf{1}, \mathbf{2}, \cdots, \mathbf{t}, t+1, t+2, \cdots \\
(FL)^1 &= \cdots, n, t, \mathbf{t-1}, \mathbf{t-2}, \cdots, \mathbf{1}, \mathbf{t+1}, t+2, t+3, \cdots \\
(FL)^2 &= \cdots, t, t+1, \mathbf{1}, \mathbf{2}, \cdots, \mathbf{t-2}, \mathbf{t-1}, \mathbf{t+2}, t+3, t+4, \cdots \\
(FL)^3 &= \cdots, t+1, t+2, \mathbf{t-1}, \mathbf{t-2}, \cdots, \mathbf{1}, \mathbf{t+2}, t+4, t+5, \cdots
\end{aligned}$$

Notice the pattern: for $k \leq n-t$, the last number in the turnstile of $(FL)^k$ is $t+k$. Thus, for $(FL)^{n-t}$, the last number in the turnstile is $t+n-t=n$. Also notice that when k is even, the first $t-1$ discs in the turnstile are in ascending order $1, 2, 3, \cdots, t-1$; when k is odd the first $t-1$ discs in the turnstile are in descending order $t-1, \cdots, 3, 2, 1$. Since n and t are both even, $n-t$ is even. We continue:

$$\begin{aligned}
(FL)^{n-t} &= ..., n-2, n-1, \mathbf{1}, \mathbf{2}, ..., \mathbf{t-2}, \mathbf{t-1}, \mathbf{n}, t, t+1, ... \\
(FL)^{n-t+1} &= ..., n-1, n, \mathbf{t-1}, \mathbf{t-2}, ..., \mathbf{2}, \mathbf{1}, \mathbf{t}, t+1, t+2, ... \\
(FL)^{n-t+1}L &= ..., n, t-1, \mathbf{t-2}, ..., \mathbf{1}, \mathbf{t}, \mathbf{t+1}, t+2, t+3, ... \\
(FL)^{n-t+1}L(FR) &= ..., n-1, n, \mathbf{t-1}, \mathbf{t+1}, \mathbf{t}, \mathbf{1}, ..., \mathbf{t-3}, t-2, t+2, ... \\
(FL)^{n-t+1}L(FRFL) &= ..., n, t-3, \mathbf{t-4}, ..., \mathbf{1}, \mathbf{t}, \mathbf{t+1}, \mathbf{t-1}, \mathbf{t-2}, t+2, ... \\
(FL)^{n-t+1}L(FRFL)^2 &= ..., n, t-5, \mathbf{t-6}, ..., \mathbf{1}, \mathbf{t}, \mathbf{t+1}, \mathbf{t-1}, .., \mathbf{t-4}, t+2, ...
\end{aligned}$$

Notice the pattern: After $(FRFL)^k$, the last number in the turnstile is $t-2k$ and the number preceding the turnstile is $t-2k-1$. Thus, after $(FRFL)^{\frac{t-2}{2}}$, the last number in the turnstile is $t-(t-2)=2$ and the number preceding the turnstile is 1. We continue:

$$\begin{aligned}
(FL)^{n-t+1}L(FRFL)^{\frac{t-2}{2}} &= ..., n, 1, \mathbf{t}, \mathbf{t+1}, \mathbf{t-1}, ..., \mathbf{3}, \mathbf{2}, t+2, ... \\
(FL)^{n-t+1}L(FRFL)^{\frac{t-2}{2}}(FR) &= ..., n-1, n, \mathbf{1}, \mathbf{2}, \cdots, \mathbf{t-1}, \mathbf{t+1}, t, t+2, ...
\end{aligned}$$

Therefore the sequence of moves $\gamma = (\delta\alpha)^{n-t+1}\alpha(\delta\beta\delta\alpha)^{\frac{t-2}{2}}(\delta\beta)$ will result in the transposition of the two adjacent discs labeled t and $t+1$.

We can transpose any two adjacent discs j and $j+1$ by first shifting the jth disc to the right most position in the turnstile, then applying γ. Since we can attain any transposition in S_n, we can attain the entire group.

Now consider the case $t=2$ and $n \geq 3$. Beginning with the start position, it is clear that any transposition of adjacent discs can be attained, so the game is solvable. Thus, Theorem 1.2 is proved.

4 Attainable Subgroups

In this section we prove Theorem 1.4.

Claim 4.1 *The attainable subgroup for the game* $[n-1, n]$, $n \geq 4$, *is* D_{2n}.

Proof: To prove this claim we analyze the relations in the group $S(n-1, n)$. Clearly the order of α is n and the order of δ is 2. Consider

$$\begin{aligned} \delta\alpha\delta &= (1, n-1)\cdots(\lfloor \tfrac{n-1}{2} \rfloor, \lfloor \tfrac{n-1}{2} \rfloor + 1)(1, 2, \ldots, n)(1, n-1)\cdots(\lfloor \tfrac{n-1}{2} \rfloor, \lfloor \tfrac{n-1}{2} \rfloor + 1) \\ &= (n, n-1, n-2, \cdots, 2, 1) = \alpha^{-1}. \end{aligned}$$

Thus, we see that $S(n-1, n) \cong D_{2n}$, and Theorem 1.4 is proved.

5 Supporting Evidence

In this section we provide some calculations which support our conjectures about the attainable subgroup for some puzzles.

The attainable subgroup, $S(t, n)$, is generated by α and δ. Using the *GAP* software, we have the following calculations for the order of $\langle \alpha, \delta \rangle$.

t	3	3	3	3	7	7	7	7	11	11	11	11
n	5	7	9	11	9	11	13	15	13	15	17	19
$\lvert\langle \alpha, \delta \rangle\rvert$	5!	7!	9!	11!	9!	11!	13!	15!	13!	15!	17!	19!

Table 1: n is odd and $t \equiv 3 \bmod 4$

t	6	6	6	6	10	10	10	10	14	14	14	14
n	9	11	13	15	13	15	17	19	17	19	21	23
$\lvert\langle \alpha, \delta \rangle\rvert$	9!	11!	13!	15!	13!	15	17!	19!	17!	19!	21!	23!

Table 2: n is odd, $t \equiv 2 \bmod 4$, and $t \leq n-2$

t	5	5	5	5	9	9	9	9	13	13	13	13
n	7	9	11	13	11	13	15	17	15	17	19	21
$\lvert\langle \alpha, \delta \rangle\rvert$	$\frac{7!}{2}$	$\frac{9!}{2}$	$\frac{11!}{2}$	$\frac{13!}{2}$	$\frac{11!}{2}$	$\frac{13!}{2}$	$\frac{15!}{2}$	$\frac{17!}{2}$	$\frac{15!}{2}$	$\frac{17!}{2}$	$\frac{19!}{2}$	$\frac{21!}{2}$

Table 3: n is odd and $t \equiv 1 \bmod 4$

Tables 1 and 2 support Conjecture 1.3, while tables 3 and 4 support conjecture 1.5.

t	4	4	4	4	8	8	8	8	12	12	12	12
n	7	9	11	13	11	13	15	17	15	17	19	21
$\|\langle\alpha,\delta\rangle\|$	$\frac{7!}{2}$	$\frac{9!}{2}$	$\frac{11!}{2}$	$\frac{13!}{2}$	$\frac{11!}{2}$	$\frac{13!}{2}$	$\frac{15!}{2}$	$\frac{17!}{2}$	$\frac{15!}{2}$	$\frac{17!}{2}$	$\frac{19!}{2}$	$\frac{21!}{2}$

Table 4: n is odd, $t \equiv 0 \bmod 4$, and $t \neq n-1$

6 Other Puzzles

Suppose another turnstile or set of turnstiles is added to the track. One can ask what the attainable subgroups are in these more general games. We give a couple of answers below.

Let $[t_1, t_2, \cdots, t_k, n]$ denote the game with n numbered discs, and k turnstiles of various sizes.

Theorem 6.1 *The game $[t_1, t_2, \cdots, t_k, n]$ is solvable if $[t_i, n]$ is solvable for some $i = 1, \cdots, k$.*

PROOF: Clearly this is true since we can leave fixed all the turnstiles except for the ith one.

Theorem 6.2 *The game $[t_1, t_2, \cdots, t_k, n]$ is not solvable when each t_i is odd and n is even.*

PROOF: Just as in case 3 of Theorem 1.1, only odd numbered discs can switch with odd numbered discs, and only evens can switch with evens. The game is not solvable and we conjecture that its attainable subgroup is A_n.

Theorem 6.2 *The game $[t_1, t_2, \cdots, t_k, n]$ is not solvable when each $t_i \equiv 0 \bmod 4$ and n is odd.*

PROOF: Let δ_i denote the permutation of S_n corresponding to the flip in the ith turnstile. As before, α will denote the shift left. Then the attainable subgroup is equal to $\langle\alpha, \delta_1, \delta_2, \cdots, \delta_k\rangle$. It is easy to see that when $t_i \equiv 0 \bmod 4$, δ_i is an even permutation. Just as in the proof of case 2 of Theorem 1.1, the attainable subgroup for the game $[t_1, t_2, \cdots, t_k, n]$ can consist only of even permutations. Thus, the game is not solvable and we conjecture that its attainable subgroup is A_n.

Theorem 6.2 *The game $[t_1, t_2, \cdots, t_k, n]$ is not solvable when each $t_i \equiv 1 \bmod 4$ and n is odd.*

PROOF: The proof of this theorem is the same as the proof above.

References

[1] C. Bennett, *TopSpin on the Symmetric Group*, Math Horizons, November 2000, 11–15.

[2] J. Wilson, *Permutation Puzzles*, College Mathematics Journal, **24** (1993), 163–165.