

# The “Flattened” Projections of Orientable Surfaces

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## INTRODUCTION

This text, produced from September 2000 through February 2001, summarizes the main points of the informal explorations into topology which I have been conducting on my own since 1999. To the best of my knowledge, the ideas are all original, and therefore much of the terminology found throughout this paper is invented (“flattening,” “non-flat deformation,” etc.). A brief glossary of these terms is included for easy reference.

No more than a basic knowledge of topology is required of the reader; I know this because “a basic knowledge of topology” is all I can boast of. Only one point deserves immediate clarification: Notation such as  $T^2 - 2D^2$  which occurs throughout the text and accompanying diagrams is read “Tee two minus two dee two,” and means a torus ( $T^2$ ) with two disks removed ( $-2D^2$ ). The superscripts denote the fact that it is surfaces being dealt with (rather than, say, three-manifolds).

This paper attempts to determine the number of different ways a given orientable surface with at least one edge can be deformed so that it rests flatly on a tabletop but is not immersed into the plane. In particular, the text examines punctured tori, and groups their various “flattenings” into two separate categories, type  $\alpha$  and type  $\beta$  flattenings.

The diagrams are more than merely illustrative; they are by far the most important aspect of this document. An understanding of the pictures is essential to understanding the text. I suggest that the reader copy a few of the images in the diagrams by hand while reading the text to get a better feel for the complex way in which the surfaces behave and relate to each other.

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**Figure 1** shows the process by which a torus with one disk removed (a “punctured torus,”  $T^2 - D^2$ ) is immersed into the plane. (A surface is considered to be immersed into the plane when all points on the surface are coplanar; this is true for step h only.) Note that, in the penultimate step (g), there is a small area in the center of the figure where two layers overlap. One of the overlapping “strips” can be said to be “on top of” the other strip, while that strip is “underneath” the first. The surface in this state is not completely immersed into the plane because “on top of” and “underneath” are meaningless concepts in two-space. Thus this distinction is lost when the surface is fully immersed in the last step. The “overlap” becomes an intersection: Both parts of the surface occupy the same space in the plane.

(Remember that, although the surface is in the form of a network of thin “strips,” these strips were originally areas of the torus, and were compressed into new shapes. The strips can always be stretched back out into their original form.)

If the process shown in the diagram is stopped at step g so that the surface remains in three-space, the distinction between “on top” and “underneath” remains. Practically speaking, the surface at this stage is “flat” in the sense that, (if we imagine it to be made of some solid, flat material such as latex), it can be laid down flat on a tabletop, or pressed between the pages of a book. Consider the real-life phenomenon of a sheet of paper lying on top of another sheet on a tabletop. Together, the two sheets approximate a segment of the plane, although we in three-space still distinguish between which sheet is on top and which is underneath.

The form of the punctured torus shown in **figure 1g** should be considered in much the same way as the sheets of paper in the preceding example. The two overlapping strips lie flat, but with distinguishable top and bottom layers. For this reason, this form will be called a “flattening” of the surface. Step f is also a flattening.

Because of these distinguishable layers, if the overlap is reversed (top and bottom are switched) as shown in the last step of **figure 2**, a new form of the surface is produced.

In **figure 2**, the deformation from step a to b and from step e to f are called “flat deformations” because the form of the surface is altered without ever “lifting” any part of

the surface off the plane or turning any part of the surface over to reveal the gray underside. In other words, the surface must remain a flattening throughout the deformation. The deformation from step b to c, c to d, and d to e all involve bringing the surface into three-space and twisting part of it to reveal the underside, since steps c and d are not flattenings. From step b to step d, one of the strips is lifted up and twisted 360°. These changes are called “non-flat deformations.” Because the process in **figure 2** involves non-flat deformations, the entire process as a whole is referred to as a non-flat deformation.

(Note: It is assumed in this text that surfaces can never intersect or pass through themselves or each other; they are treated as though they were solid substances.)

The first step and the last step of **figure 2** are called “distinct” flattenings of the punctured torus because it is impossible to deform one into the other without a non-flat deformation; one cannot be flatly deformed into the other. (This fact will be demonstrated later.) The forms in the first two steps in the diagram are considered “indistinct” flattenings of the punctured torus because a non-flat deformation is unnecessary for deforming one into the other; each can be flatly deformed into the other.

**Figure 3** shows three examples of flat deformations of the punctured torus. Every form in the diagrams is indistinct from the others.

A more precise definition of flat versus non-flat deformations involves vectors. A deformation is non-flat if and only if, at some step during the deformation process, there exists a vector normal to the surface at some point which is parallel to the plane that the original flattening had been considered to be “lying upon.” Such a normal vector exists in steps c and d of **figure 2**, radiating from the vertical sections of the elevated gray hoops. In a flat deformation, the normal vectors to the surface constantly remain approximately perpendicular to this imagined plane, as in steps a, b, e, and f of **figure 2**. (We say “approximately perpendicular” because, at the regions where the surface overlaps itself, the surface must deviate slightly from the plane; if this were not the case the surface would become fully immersed in the plane, as in **figure 1h**.) This definition was first suggested by Prof. Douglas Ravenel of Rochester University, NY.

The single-punctured torus is not the only surface which can be flattened in this manner. In fact, each of the above definitions applies to every orientable surface with at least one edge. **Figure 4** shows two distinct flattenings of a three-punctured sphere ( $S^2-3D^2$ ) and three distinct flattenings of a two-punctured torus ( $T^2-2D^2$ ). **Figure 5** shows three distinct flattenings of a single-punctured double-torus ( $2T^2-D^2$ ). The

flattenings of  $n$ -punctured double-tori are extremely varied and interesting, but will not be discussed further here. This text will deal primarily with  $n$ -punctured tori ( $T^2 - nD^2$ ).

Note how, in the diagram, the number of disks removed from the figures in the first steps is equal to the number of edges of each of their flattenings and of each intermediate step. In other words, the number of edges remains constant throughout the deformation process.

Note that the overlap in the third flattening of the two-punctured torus at the bottom of **figure 4** can be reversed by simply rotating the figure  $90^\circ$ . To reverse the overlap of the second flattening of the three-punctured sphere, a flat deformation similar to that shown in **figure 12** is used. (This deformation will be discussed later.)

The most important pattern that this text will examine is the number of distinct flattenings that can be produced from a given surface, in this case the  $n$ -punctured torus. Finding this pattern first involves finding a method for determining whether two flattenings of a surface are distinct or indistinct. When the forms are indistinct, the method is simple: Just find a sequence of steps whereby one of the forms is flatly deformed into the other. When the forms are distinct, however, verification is much more difficult. At this time, no formal, fail-safe procedure for doing this has yet been discovered (although a method of recognizing distinct flattenings of  $n$ -punctured tori will be a major topic of this text). In general, a process of trial and error can be used to suggest that two flattenings are distinct (after numerous trials, if the forms have still not shown themselves to be indistinct, it is assumed that they are not indistinct). For example, suppose we wanted to show that the two flattenings of the single-punctured torus are distinct. Refer back to **figure 3**. The first two deformations pictured in that diagram can put loops into the strips of the surface but cannot reverse the overlap in the center; the third deformation pictured can reverse the overlap but adds two inextricable loops to one of the strips. None of these three flat deformations, nor any variation and/or combination thereof, can ever possibly turn the form of **figure 2a** into the form of **figure 2f**. Only the non-flat deformation of **figure 2** can accomplish that. Again, the reader is encouraged to try this him/herself in order to better understand why these two flattenings of the punctured torus are distinct.

Flattenings of  $n$ -punctured tori can be divided into two categories, type  $\alpha$  and type  $\beta$  flattenings. Type  $\beta$  flattenings are more numerous and complex than the  $\alpha$  forms, but since they only occur for tori with three punctures or more, they will be discussed later. Type  $\alpha$  flattenings occur for all tori with one puncture or more. Their derivations are simpler than those of type  $\beta$  flattenings, so they will be treated first.

**Figure 6** shows the process by which a four-punctured torus is flattened. From

the diagram, one can draw the generalized conclusion that a torus with  $n$  punctures can be flattened into a circular strip, or “ring,” with  $n$  small “ringlets” attached to it at various spots along its length. Each of these ringlets can be pointing either clockwise or counter clockwise along the large, circular ring. The final, simplified form is dependent on the sequence of the directions that the ringlets are pointing in. (The term “simplified,” as it is used here, will be precisely defined momentarily.) For the four-punctured torus, there exist six of these sequences, all of which appear at the bottom of the diagram. Some, but not all, of these six flattenings are distinct. Forms #1, #2, and #3 can be flatly deformed into each other as shown in **figure 7**, so they are indistinct flattenings.

The term “simplified forms” will now be defined. Any flattening overlaps itself at a certain number of places. In the ring-and-ringlets forms of an  $n$ -punctured torus, there are  $n$  areas of overlap. The simplified forms in a set of indistinct flattenings are those forms with the fewest possible areas of overlap. To simplify a flattening means to minimize the number of areas of overlap.

(Flattenings of punctured multi-tori as in **figure 5** can have more than one simplified form, although only one is shown for each of the three distinct flattenings in the diagram. This is not true for punctured single-tori, so it is not discussed further here.)

The six flattenings at the bottom of **figure 6** are of the type  $\alpha$  variety because they can be shaped like rings and ringlets. These particular forms are called ring-and-ringlet forms of the flattenings. The four-punctured torus has exactly four distinct type  $\alpha$  flattenings. Their simplified forms appear on the right side of **figure 8**, next to the ring-and-ringlets forms from which they are derived. Although **figure 8** does not show the complete deformation processes (due to space constraints), an intermediate step is given for each deformation, and **figure 9** shows the complete process for form #4. For the other flattenings, the reader should try to work out the full, detailed processes on his/her own. (A hint: In the intermediate steps shown for each flattening except #4, there are a number of long, C-shaped interspaces between the enlarged ringlets. These interspaces can be shrunk into the small circular holes which appear in the simplified forms. Small circular holes like these will be called SCHs.)

As the number of punctures  $n$  increases, the number of distinct type  $\alpha$  flattenings,  $J$ , increases. To make the relation between  $J$  and  $n$  clear, **figure 10** shows the simplified form of each distinct type  $\alpha$  flattening for tori of one, two, three, four, five, and six punctures. Each row contains flattenings of one surface. The vertical columns contain flattenings with similar qualities. For example, the flattenings in the first two

columns had all their ringlets pointing in the same direction (clockwise for column one, counter clockwise for column two). Each flattening is equal to the one above it, with an SCH added to the surface. The reader might want to examine the ring-and-ringlet forms (not shown) of the flattenings of each column for common traits. We will refer back to this figure frequently later on.

Judging from **figure 10**, one can conclude that when  $n$  increments from an odd value to an even value,  $J$  increases by one, and when  $n$  increments from an even value to an odd value,  $J$  stays the same. Thus a formula for  $J$  in terms of  $n$  would be:

$$J = (n + 3) / 2 \quad (n \text{ odd})$$

$$J = (n + 4) / 2 \quad (n \text{ even})$$

This formula has been shown to hold true for all values of  $n$ . To see why it works, refer back to **figure 8**, in which the four simplified forms of the four-punctured torus are produced. In simplifying the first three ring-and-ringlets forms, two of the ringlets “cancel out” in each case, leaving two SCHs and just two ringlets pointing opposite directions, which simplify into the four-chambered cruciform that appears at the top of the final simplified form. This happens for any ring-and-ringlets form (for any value of  $n$ ) in which the sequence of ringlets is such that  $x$  consecutive ringlets are pointing clockwise and  $y$  consecutive ringlets are pointing counterclockwise, where  $x$  and  $y$  are both at least one and  $x + y = n$ . This is true of all forms in the third column of **figure 10**.

In the fourth ring-and-ringlets form, the directions of the ringlets alternate, so that none of the ringlets “cancel” another out as they do in the other forms. The simplified form of this flattening is different from the simplified forms of the other five; in other words, this flattening is unique. In fact, any ring-and-ringlets form in which the directions of the ringlets alternate is distinct from other ring-and-ringlets forms of the same surface. Because this alternating sequence is possible only when  $n$  is even,  $J$  increments when  $n$  increments from an odd value to an even value due to the addition of this unique, distinct flattening. Unlike the other type  $\alpha$  flattenings of the surface, this new flattening cannot be produced by just putting an SCH into any of the simplified forms of the surface with  $n-1$  edges.

The simplified form of #4 appears in the fourth column of **figure 10**. The ring-and-ringlets forms of all the flattenings in this column have this property in common: The ringlets can be divided into four sets of consecutive ringlets, with at least one ringlet per set. In each set all of the ringlets point in the same direction, and the

direction is different in each pair of adjacent sets.

**Figure 11** shows the simplified forms of the type  $\alpha$  flattenings with alternating ringlets for tori of two, four, six, and eight punctures. As the diagram makes clear, each of these forms is equal to the form above it (with  $n-2$  edges) plus a formation that looks like an upside-down “mushroom” growing in a “window.” (“Mushroom” is the name for the little loop attached to the main form by a stem that looks like a mushroom.) The number of mushrooms that appear in the simplified form of a type  $\alpha$  flattening with alternating ringlets is  $(n/2)-1$ . Note where the simplified forms in this diagram are placed in **figure 10**.

In the fifth and sixth forms of **figure 8**, all but one of the ringlets “cancel” each other out, leaving just that one ringlet and three SCHs. This one remaining ringlet points clockwise in the fifth form and counter clockwise in the sixth form, so the two flattenings are distinct. (Compare these two forms with **figure 2a** and **2f**.) This occurs for all ring-and-ringlets forms in which all of the ringlets are pointing in the same direction (clockwise or counter clockwise). Since this is a possibility no matter what the value of  $n$  is, there are always at least two distinct type  $\alpha$  flattenings of an  $n$ -punctured torus.

Here is the recursive rule for finding all simplified type  $\alpha$  flattenings of a torus with  $n$  punctures. When  $n$  is an odd number greater than 1, the set of simplified type  $\alpha$  flattenings consists of each element of that set for a torus with  $n-1$  punctures, with an SCH added to the surface. When  $n$  is even, the set also includes one additional flattening, derived from the ring-and-ringlets form with alternating ringlets. Study **figure 10** to verify this rule.

We now have the correct formula for determining the exact number  $J$  of distinct type  $\alpha$  flattenings of a  $n$ -punctured torus. As the formula makes clear, the function  $J = f(n)$  is essentially linear; the rate at which  $J$  increases with respect to  $n$  alternates between 1 and 0  $J/n$ , with an average rate of 0.5  $J/n$ .

This is not the case with type  $\beta$  flattenings, however. Type  $\beta$  flattenings are not derived from ring-and-ringlets forms, but from the simplified type  $\alpha$  flattenings themselves. As the value of  $n$  increases, the number  $K$  of type  $\beta$  flattenings tends to zoom skyward at a rapidly increasing rate. To produce a type  $\beta$  flattening, we take one of the SCHs which occur in certain simplified type  $\alpha$  flattenings and turn it “inside out” in another part of the figure, as shown in **figure 12**. This produces another mushroom, like those which appear in certain type  $\alpha$  flattenings. (Compare **figure 12** with **figure 4-I**, which shows a flattening of a three-punctured sphere and contains a mushroom. This is analogous to a type  $\beta$  flattening for a sphere.) Because the mushroom is present in a

different *part* of the surface than it is with the type  $\alpha$  flattenings, these flattenings are distinct from all of the type  $\alpha$  flattenings of the same surface. Unlike type  $\alpha$  flattenings, type  $\beta$  flattenings cannot be flatly deformed into ring-and-ringlets forms.

Mushrooms, whether appearing as part of a type  $\alpha$  or  $\beta$  flattening, can always be flatly deformed so that the overlap of the loop (the mushroom's "cap") and stem is reversed (top and bottom layers are switched), as shown in **figure 13**. Because of this property, a flattening containing any number of mushrooms is indistinct from the same form but with some or all of its mushrooms reversed. (The property can also be used to show that for any flattening of any orientable surface at all, the result of flipping the form over like a pancake can also be achieved through a flat deformation. This matter will not be treated further here.)

There are two places in a type  $\alpha$  flattening where an SCH may be turned inside-out in this manner to produce a type  $\beta$  flattening, and both of these possibilities are exemplified in **figure 12**. The example on the left (I) shows the first possibility: the mushroom is formed inside the four-chambered cruciform. The example on the right (II) shows the other possibility: the mushroom is formed right next to another preexisting mushroom. Any combination of any number of these two possibilities will result in a type  $\beta$  flattening.

Type  $\beta$  flattenings can be derived from any type  $\alpha$  flattening unless its ring-and-ringlets form has all of the ringlets pointing in the same direction. These type  $\alpha$  flattenings appear in the first and second vertical columns of **figure 10**. None of the forms in these columns contains either a four-chambered-cruciform or any preexisting mushrooms. Therefore it is impossible to produce a type  $\beta$  flattening by deforming an SCH into a mushroom anywhere on one of these type  $\alpha$  flattenings. If this is done, then the resulting form will be indistinct from the form in the third column of **figure 10** (and, of course, in the same row as the original). It doesn't matter where the mushroom is formed: It can be formed along the outer edge of the flattening (or inside the pretzel-like loop on the top of the form, which is the same thing), or it can be formed inside another SCH. Either way, the result will be the same.

**Figure 14** proves that this is so, using a four-punctured torus to demonstrate. We start off at step a with the type  $\alpha$  flattening from the first column of **figure 10**. In the first five steps (a through b), one of the SCHs is turned inside-out to form a mushroom inside another SCH. (This is the only part of the sequence involving a non-flat deformation, so all the forms after this are indistinct.) The fourteenth step, marked "c", is the form which would have resulted had the mushroom been formed along the outer edge of the original, instead of inside another SCH. The eighteenth and final form,



marked “d”, is the form which appears in the third column of **figure 10**.

When the original type  $\alpha$  flattening is not from the first or second column of **figure 10** (as we will assume henceforth), then a type  $\beta$  flattening can be produced by turning an SCH into a mushroom in one of the two aforementioned possible places (inside a cruciform or next to another mushroom, as shown in **figure 12**; this is provided that the original flattening has at least one SCH to be deformed into a mushroom).

If an SCH is deformed into a mushroom anywhere other than the two aforementioned possibilities, the mushroom will degenerate into some simpler formation via a flat deformation. If the mushroom is formed along the outer edge, then the mushroom will revert back into an SCH and the flattening will remain unchanged. In other words, the original form is indistinct from the final form, even though a non-flat deformation was used. The same is true when a mushroom is formed on the edge of the “cap” of a preexisting mushroom. The reader should try to demonstrate these facts.

Also, a type  $\beta$  flattening cannot be produced from a type  $\alpha$  flattening by turning an SCH into a mushroom inside another SCH; doing this only produces a different type  $\alpha$  flattening. This new flattening is distinct from the original flattening, and would appear in **figure 10** directly to the right of the original.

**Figure 15** shows the simplified forms of each distinct type  $\beta$  flattening for tori of one, two, three, four, five, and six punctures. (Unlike **figure 10**, this diagram is not organized into vertical columns each containing similar flattenings.) In rows 5 and 6, not all of the forms were derived from the same type  $\alpha$  flattening: Some are based on the corresponding form from the third column of **figure 10**, while others are based on the form in the fourth column. (These are the rightmost two forms from row 5 and the five lower forms from row 6.)

Note that, in the diagram, all of the flattenings are bounded by a continuous, uninterrupted edge. This is a property of the simplified forms of all type  $\beta$  flattenings, and also of all type  $\alpha$  flattenings not appearing in the first two columns of **figure 10**.

As the **figure 15** makes clear, the number  $K$  of distinct type  $\beta$  flattenings increases at an ever-accelerating rate as  $n$  increases. Although no equation has yet been developed which relates  $K$  to  $n$ , there is a systematic method of listing type  $\beta$  flattenings without drawing every one. Suppose we wanted to quickly determine how many distinct type  $\beta$  flattenings exist for a six-punctured torus. Referring to the sixth row of **figure 10**, we see that only the form in the third and fourth column can be used to derive type  $\beta$  flattenings, because the form in the fifth column has no SCHs and the forms in the first and second columns can never be used, as explained above. Four type  $\beta$  flattenings can be derived from the form in the third column because we have a

choice of turning one, two, three, or all four of the SCHs into mushrooms inside the cruciform. Five type  $\beta$  flattenings can be derived from the form in the fourth column. Our choices here are: one mushroom in the cruciform; two mushrooms in the cruciform; one mushroom next to the preexisting mushroom; two mushrooms next to the preexisting mushroom; or one mushroom in the cruciform and one next to the preexisting mushroom. No other possibilities exist, so the total number of distinct type  $\beta$  flattenings, then, is  $4+5=9$ . This is in agreement with the sixth row of **figure 15**.

Careful listing and counting has yielded the values of  $K$  for values of  $n$  as high as 10. The following table charts  $n$  versus  $K$ . The reader is encouraged to verify these values using the listing method described above.

$n$ :	1	2	3	4	5	6	7	8	9	10
$K$ :	0	0	1	2	5	9	16	26	41	62

The total number of distinct flattenings of an  $n$ -punctured torus is  $J + K$ . This number is what we set out to find in the first place. The following table charts  $n$  versus  $J + K$  for values of  $n$  up to 10.

$n$ :	1	2	3	4	5	6	7	8	9	10
$J + K$ :	2	3	4	6	9	14	21	32	47	69

We have now developed a foolproof method of determining the exact number of distinct flattenings of an  $n$ -punctured torus. With that, this text concludes. A sequel (if one comes to exist) will discuss flattenings of spheres and the connective sum of multiple tori (see **figures 4 & 5** for a preview), the reversibility of the front and back sides of a flattening, and (hopefully) the derivation of an explicit formula relating  $n$  and  $K$ . Thank you for attending to my fancies.

## REFERENCE

Weeks, Jeffrey R., *The Shape of Space: How to Visualize Surfaces and Three-Dimensional Manifolds*, Marcel Dekker, Inc., New York, 1985  
 Recommended as an introductory text on topology.

## ESSENTIAL TERMS:

**puncture:** The hole left by a disk removed from a surface; or, more precisely, the edge of that hole.

**overlap:** In a flattening, any area where one part of the surface covers another.

**flattening:** A projection of an orientable surface which can lie flat on a tabletop but is not immersed into the plane.

**flat deformation:** A deformation of a flattening, in which the surface at no time ceases to be a flattening.

**non-flat deformation:** Any deformation which is not a flat deformation (i.e., the surface at some stage of the process is not a flattening).

**distinct flattenings:** Flattenings which cannot be deformed into one another via a flat deformation.

**indistinct flattenings:** Flattenings which can be deformed into one another via a flat deformation.

**simplified form:** Any form of a given flattening which has a minimal number of overlaps.

**type  $\alpha$  flattenings:** A flattening of an  $n$ -punctured torus which can be flatly deformed into a ring-and-ringlets form.

**ring-and-ringlets form:** A form of a type  $\alpha$  flattening which is characterized by a wide, circular ring that has small ringlets attached to it along its length, as shown in figures 6, 8, etc.

**SCH:** A puncture in the form of a small circular hole.

**cruciform:** The four-chambered formation found in many flattenings, bounded by a single edge which crosses over itself at four points, as shown in the last step of figure 4-II, etc.

**mushroom:** A formation found in many flattenings, characterized by a loop whose inner edge is connected to the edge of the surface by a strip, as shown in step one of figure 13, etc.

**type  $\beta$  flattenings:** Any flattening of an  $n$ -punctured torus which cannot be deformed into a ring-and-ringlets form.

## VARIABLES:

$n$  = number of punctures in torus

$J$  = number of distinct type  $\alpha$  flattenings of surface

$K$  = number of distinct type  $\beta$  flattenings of surface