Fuzzy Topological Spaces

Part I (May 10)

Fuzzy Sets and Fuzzy Topologies: Early Ideas and Obstacles

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Example of an imprecise or “fuzzy” description of a set of numbers:

“The set of real numbers much greater than 10.”

L. A. Zadeh (1965): A fuzzy set A in a given set X is associated with an assignment of a degree of membership in A to each point of X where degree of membership means some real number from the closed interval $[0, 1]$. A larger degree of membership reflects a stronger sense of “belonging” to A.

Definition. For a set X we define a *fuzzy set* in X to be a function $\mu : X \to [0, 1]$.

Here $\mu(x)$ “represents the degree of membership of x in the fuzzy set A.”

Note. Any *subset* A of a set X can be identified with its characteristic function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

and such characteristic functions are fuzzy sets in X. Thus, **fuzzy sets generalize subsets**.

Definition. The characteristic functions of subsets of a set X are referred to as the *crisp* fuzzy sets in X.

Our Example. "The fuzzy set of real numbers much greater than 10" is a fuzzy set in $\mathbb{R}$ that could be described by the continuous function

$$\mu : \mathbb{R} \to [0,1] \text{ given by}$$

$$\mu(x) = \begin{cases} 0 & \text{if} \quad x \leq 10 \\ \frac{x-10}{90} & \text{if} \quad 10 < x < 100 \\ 1 & \text{if} \quad x \geq 100 \end{cases}$$

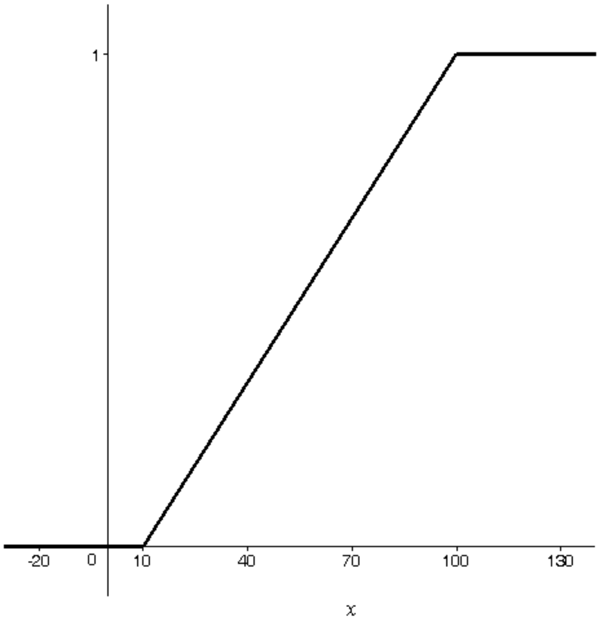

Figure 1. One fuzzy set of "real numbers much greater than 10"

Let μ and ν be $[0,1]$-valued functions defined on a fixed set X (i.e., fuzzy sets in X). Write

$$\mu \leq \nu \text{ to mean that } \mu(x) \leq \nu(x) \text{ for all } x \in X, \text{ and}$$

$$\mu = \nu \text{ to mean that } \mu(x) = \nu(x) \text{ for all } x \in X.$$

The *maximum* function $\mu \vee \nu$, the *minimum* function $\mu \wedge \nu$, and the *complement* function $1 - \mu$ defined on X by the rules

$$\begin{aligned}
(\mu \vee \nu)(x) &= \max\{\mu(x), \nu(x)\}, \\
(\mu \wedge \nu)(x) &= \min\{\mu(x), \nu(x)\}, \text{ and} \\
(1 - \mu)(x) &= 1 - \mu(x),
\end{aligned}$$

respectively, are $[0,1]$-valued functions.

So $\mu \vee \nu$, $\mu \wedge \nu$, and $1 - \mu$ are fuzzy sets in X if μ and ν are.

For **subsets** A and B of X,

$$\begin{gathered} A = B \text{ if and only if } \chi_A = \chi_B, \\ A \subseteq B \text{ if and only if } \chi_A \leq \chi_B, \\ \chi_{A \cup B} = \chi_A \vee \chi_B, \\ \chi_{A \cap B} = \chi_A \wedge \chi_B, \text{ and} \\ \chi_{A^c} = 1 - \chi_A \ . \end{gathered}$$

Definition. For two fuzzy sets μ and ν in X:

μ and ν are *equal* if and only if $\mu = \nu$,
μ is *contained in* ν if and only if $\mu \leq \nu$,
the *union* of μ and ν is $\mu \vee \nu$,
the *intersection* of μ and ν is $\mu \wedge \nu$, and
the *complement* of μ is $1 - \mu$.

Theorem. For fuzzy sets μ, μ_1, μ_2, and ν in X:

$$\begin{aligned} &\mu = \nu \text{ if and only if } \mu \leq \nu \text{ and } \mu \geq \nu, \\ &\nu \wedge (\mu_1 \vee \mu_2) = (\nu \wedge \mu_1) \vee (\nu \wedge \mu_2), \\ &\nu \vee (\mu_1 \wedge \mu_2) = (\nu \vee \mu_1) \wedge (\nu \vee \mu_2), \\ &1 - (\mu_1 \vee \mu_2) = (1 - \mu_1) \wedge (1 - \mu_2), \text{ and} \\ &1 - (\mu_1 \wedge \mu_2) = (1 - \mu_1) \vee (1 - \mu_2). \end{aligned}$$

Definition. The *union* (respectively, *intersection)* of the fuzzy sets

$$\mu_i \ (i \in I)$$

is defined by

$$\bigvee_{i \in I} \mu_i(x) = \sup\{\mu_i(x) : i \in I\}$$

$$\text{(respectively, } \bigwedge_{i \in I} \mu_i(x) = \inf\{\mu_i(x) : i \in I\}).$$

(If the μ_i's are crisp, these suprema and infima are actually maxima and minima.)

Theorem. For fuzzy sets μ_i $(i \in I)$:

$$\begin{aligned}
\nu \wedge (\bigvee_{i \in I} \mu_i) &= \bigvee_{i \in I} (\nu \wedge \mu_i), \\
\nu \vee (\bigwedge_{i \in I} \mu_i) &= \bigwedge_{i \in I} (\nu \vee \mu_i), \\
1 - \bigvee_{i \in I} \mu_i &= \bigwedge_{i \in I} (1 - \mu_i), \text{ and} \\
1 - \bigwedge_{i \in I} \mu_i &= \bigvee_{i \in I} (1 - \mu_i).
\end{aligned}$$

Recall. A *topology* on a set X is a collection τ of subsets of X that

(i) contains $\varnothing$ and X,

(ii) is closed under formation of finite intersections and,

(iii) is closed under formation of arbitrary unions.

The pair (X, τ) is called a *topological space.*

Definition. (C.L. Chang, 1968) A *fuzzy topology* on a set X is a collection δ of fuzzy sets in X satisfying:

(i) $0 \in \delta$ and $1 \in \delta$,

(ii) if μ and ν belong to δ, then so does $\mu \wedge \nu$, and

(iii) if μ_i belongs to δ for each $i \in I$, then so does $\bigvee\limits_{i \in I} \mu_i$.

If δ is a fuzzy topology on X, then the pair (X, δ) is called a *fuzzy topological space.*

Members of δ are called *open* fuzzy sets.

Fuzzy sets of the form $1 - \mu$, where μ is an open fuzzy set, are called *closed* fuzzy sets.

Examples of fuzzy topologies:

Any topology on a set X (subsets are identified with their characteristic functions)

The indiscrete fuzzy topology $\{0, 1\}$ on a set X (= indiscrete topology on X)

The *discrete* fuzzy topology on X containing all fuzzy sets in X

The collection of all crisp fuzzy sets in X (= discrete topology on X)

The collection of all constant fuzzy sets in X

The intersection of any family of fuzzy topologies on a set X

Let (X, τ) be a topological space.

Recall that a function $f : X \to \mathbb{R}$ is *lower-semicontinuous* (*l.s.c.*) if and only if for all $\alpha \in \mathbb{R}$, $\{x \in X : f(x) > \alpha\}$ is an open set.

- Continuous real-valued functions on X are l.s.c.

  (In particular, any constant real-valued function on X is l.s.c.)

- The supremum of any family of $[0, 1]$-valued l.s.c. functions on X is also l.s.c.

- The infimum of any finite family of $[0, 1]$-valued l.s.c. functions on X is also l.s.c.

Definition. For a topological space (X, τ), the *l.s.c. fuzzy topology* on X associated with τ is

$$\omega(\tau) = \{\mu : X \to [0, 1] : \mu \text{ is l.s.c.}\}$$

Functions and Fuzzy Continuity

Let X and Y be sets, and let $f : X \to Y$ be a function.

For a fuzzy set ν in Y, the *inverse image* of ν under f is the fuzzy set $f^{-1}(\nu)$ in X by the rule

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for } x \in X.$$

(I.e., $f^{-1}(\nu) = \nu \circ f$.)

For a fuzzy set μ in X, the image of μ under f is the fuzzy set $f(\mu)$ in Y defined, for $y \in Y$, by the rule

$$f(\mu)(y) = \begin{cases} \sup\{\mu(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing \end{cases}.$$

These definitions preserve their analogues in the non-fuzzy setting.

The usual properties relating images and inverse images of subsets to unions, intersections, and complements also hold for fuzzy sets.

Definition. Given fuzzy topological spaces (X, δ) and (Y, γ), a function $f : X \to Y$ is *fuzzy continuous* if the inverse image under f of any open fuzzy set in Y is an open fuzzy set in X; i.e., if $f^{-1}(\nu) \in \delta$ whenever $\nu \in \gamma$.

Proposition. (a) The identity mapping $id_X : (X, \delta) \to (X, \delta)$ on a fuzzy topological space (X, δ) is fuzzy continuous.

(b) A composition of fuzzy continuous functions is fuzzy continuous.

Proof. (a) For $\nu \in \delta$, $id_X^{-1}(\nu) = \nu \circ id_X = \nu$.

(b) Let $f : (X, \delta) \to (Y, \gamma)$ and $g : (Y, \gamma) \to (Z, \beta)$ be fuzzy continuous. For $\eta \in \beta$,

$$\begin{aligned} (g \circ f)^{-1}(\eta) &= \eta \circ (g \circ f) \\ &= (\eta \circ g) \circ f \\ &= f^{-1}(\eta \circ g) \\ &= f^{-1}(g^{-1}(\eta)). \end{aligned}$$

$g^{-1}(\eta) \in \gamma$ since g is fuzzy continuous, and so $(g \circ f)^{-1}(\eta) = f^{-1}(g^{-1}(\eta)) \in \delta$ since f is fuzzy continuous.

Definition. A fuzzy topological space (X, δ) is *compact* if every cover of X by members of δ contains a finite subcover; i.e, if $\mu_i \in \delta$ for every $i \in I$ and $\bigvee_{i \in I} \mu_i = 1$, then there are finitely many indices $i_1, i_2, \ldots, i_n \in I$ such that $\bigvee_{j=1}^{n} \mu_{i_j} = 1$.

Theorem. *Let (X, δ) and (Y, γ) be fuzzy topological spaces, with (X, δ) compact, and let $f : X \to Y$ be a fuzzy continuous surjection. Then (Y, γ) is also compact.*

Proof. Let $\nu_i \in \gamma$ for each $i \in I$ and assume that $\bigvee_{i \in I} \nu_i = 1$. For each $x \in X$, $\bigvee_{i \in I} f^{-1}(\nu_i)(x) = \bigvee_{i \in I} \nu_i(f(x)) = 1$, so the δ-open fuzzy sets $f^{-1}(\nu_i)$ $(i \in I)$ cover X. Thus, for finitely many indices $i_1, i_2, \ldots, i_n \in I$, $\bigvee_{j=1}^{n} f^{-1}(\nu_{i_j}) = 1$. If ν is any fuzzy set in Y, the fact that f is a surjection mapping onto Y implies that, for any $y \in Y$,

$$\begin{aligned} f(f^{-1}(\nu))(y) &= \sup\{f^{-1}(\nu)(z) : z \in f^{-1}(y)\} \\ &= \sup\{\nu(f(z)) : f(z) = y\} \\ &= \nu(y) \end{aligned}$$

so that $f(f^{-1}(\nu)) = \nu$. Thus, as fuzzy sets in Y,

$$1 = f(1) = f\Big(\bigvee_{j=1}^{n} f^{-1}(\nu_{i_j})\Big) = \bigvee_{j=1}^{n} f(f^{-1}(\nu_{i_j})) = \bigvee_{j=1}^{n} \nu_{i_j} .$$

Therefore, (Y, γ) is compact.

Products of Fuzzy Topological Spaces

Definition. (a) Let (X_i, δ_i) be a fuzzy topological space for each index $i \in I$. The *product fuzzy topology* $\delta = \prod\limits_{i \in I} \delta_i$ on the set $X = \prod\limits_{i \in I} X_i$ is the coarsest fuzzy topology on X making all the projection mappings $\pi_i : X \to X_i$ fuzzy continuous.

(b) A *base* for a fuzzy topological space (X, δ) is a subcollection $\mathcal{B}$ of δ such that each member μ of δ can be written as $\mu = \bigvee\limits_{\alpha \in \Lambda} \mu_\alpha$, where each μ_α belongs to $\mathcal{B}$.

(c) A *subbase* for (X, δ) is a subcollection $\mathcal{S}$ of δ such that the collection of infima of finite subfamilies of $\mathcal{S}$ forms a base for (X, δ).

Theorem. A subbase for the product fuzzy topology on $(X, \delta) = (\prod\limits_{i \in I} X_i, \prod\limits_{i \in I} \delta_i)$ is given by $\mathcal{S} = \{\pi_i^{-1}(\mu_i) : \mu_i \in \delta_i \ , \ i \in I\}$ so that a base can be taken to be $\mathcal{B} = \{\bigwedge\limits_{j=1}^{n} \pi_{i_j}^{-1}(\mu_{i_j}) : \mu_{i_j} \in \delta_{_{i_j}} \ , i_j \in I, j = 1...n, n \in \mathbb{N}\}$.

Two-factor case: Basic open fuzzy sets on $(X_1 \times X_2, \delta_1 \times \delta_2)$ are of the form $\pi_1^{-1}(\mu_1) \wedge \pi_2^{-1}(\mu_2)$, e.g., "$\mu_1 \times \mu_2$." For indeed, if $x \in X_1$ and $y \in X_2$, then

$$\begin{aligned}
(\mu_1 \times \mu_2)(x, y) &= (\pi_1^{-1}(\mu_1) \wedge \pi_2^{-1}(\mu_2))(x, y) \\
&= ((\mu_1 \circ \pi_1) \wedge (\mu_2 \circ \pi_2))(x, y) \\
&= \mu_1(\pi_1(x, y)) \wedge \mu_2(\pi_2(x, y)) \\
&= \mu_1(x) \wedge \mu_2(y) \ .
\end{aligned}$$

Lemma (***Alexander Subbase Theorem***). (Goguen, 1973) *If $\mathcal{S}$ is a subbase for a fuzzy topological space (X, δ), then (X, δ) is compact if and only if every cover of X by members of $\mathcal{S}$ has a finite subcover (i.e., if $\mu_\alpha \in \mathcal{S}$ for each $\alpha \in \Lambda$ and $\bigvee_{\alpha \in \Lambda} \mu_\alpha = 1$, then there are finitely many indices α_i $(i = 1, ..., n)$ such that $\bigvee_{i=1}^{n} \mu_{\alpha_i} = 1$).*

Finite Fuzzy Tychonoff Theorem (Goguen, 1973). *Let n be a positive integer and, for each $i = 1, ..., n$, let (X_i, δ_i) be a compact fuzzy topological space. Then $(X, \delta) = (\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \delta_i)$ is compact.*

Proof. We will say that a collection of open fuzzy sets of a fuzzy topological space has the *finite union property* (*FUP*) if none of its finite subcollections cover the space (i.e., none of its finite subcollections have supremum identically equal to 1). Since $\mathcal{S} = \{\pi_i^{-1}(\mu_i) : \mu_i \in \delta_i \ , \ i = 1 \ ... \ n\}$ is a subbase for (X, δ), by the lemma it suffices to show that no subcollection of $\mathcal{S}$ with FUP covers X. Let $\mathcal{C}$ be a subcollection of $\mathcal{S}$ with FUP. For each $i = 1, ..., n$, let $\mathcal{C}_i = \{\mu \in \delta_i : \pi_i^{-1}(\mu) \in \mathcal{C}\}$. Then $\mathcal{C}_i$ is a collection of open fuzzy sets in (X_i, δ_i) with FUP. Indeed, if $\mu_{i,1}, \ \mu_{i,2}, \ ... \ , \ \mu_{i,k} \in \mathcal{C}_i$ satisfy $\bigvee_{j=1}^{k} \mu_{i,j} = 1_{X_i}$, then

$$\bigvee_{j=1}^{k} \pi_i^{-1}(\mu_{i,j}) = \pi_i^{-1}(\bigvee_{j=1}^{k} \mu_{i,j}) = \pi_i^{-1}(1_{X_i}) = 1_X \ ,$$

and this would contradict the fact that $\mathcal{C}$ has FUP.

Therefore, by the compactness of (X_i, δ_i), the collection $\mathcal{C}_i$ cannot cover X_i, and we can select a point $x_i \in X_i$ such that $(\bigvee \mathcal{C}_i)(x_i) = a_i < 1$. Now if we consider the point $x = (x_1, x_2, ..., x_n) \in X$ and the collection $\mathcal{C}_i' = \{\pi_i^{-1}(\mu) : \mu \in \delta_i\} \cap \mathcal{C}$, then it follows that

$$\begin{aligned}(\bigvee \mathcal{C}_i')(x) &= \bigvee \{\pi_i^{-1}(\mu)(x) : \mu \in \delta_i \text{ and } \pi_i^{-1}(\mu) \in \mathcal{C}\} \\ &= \bigvee \{\mu(x_i) : \mu \in \delta_i \text{ and } \pi_i^{-1}(\mu) \in \mathcal{C}\} \\ &= (\bigvee \mathcal{C}_i)(x_i) \\ &= a_i \ .\end{aligned}$$

Further, noting that $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i'$, we obtain

$$(\bigvee \mathcal{C})(x) = \bigvee_{i=1}^{n} (\bigvee \mathcal{C}_i')(x) = \bigvee_{i=1}^{n} (\bigvee \mathcal{C}_i)(x_i) = \bigvee_{i=1}^{n} a_i$$

which is strictly less than 1 since each of the *finitely many* real numbers a_i is strictly less than 1. Thus, $\bigvee \mathcal{C} \neq 1$, as desired.

Example. A non-compact product of countably many compact fuzzy topological spaces.

For each positive integer i let $X_i = \mathbb{N}$, the set of positive integers, let μ_i be the constant fuzzy set in $\mathbb{N}$ given by $\mu_i(x) = \frac{i-1}{i}$ for $x \in \mathbb{N}$, and let $\delta_i = \{0, \mu_i, 1\} \cup \{\mu_i \chi_{\{1,2,\ldots,n\}} : n \in \mathbb{N}\}$. Here note that the function $\mu_i \chi_{\{1,2,\ldots,n\}}$ is a product; its graph is shown in Figure 2 for the case where $i = 6$ and $n = 10$.

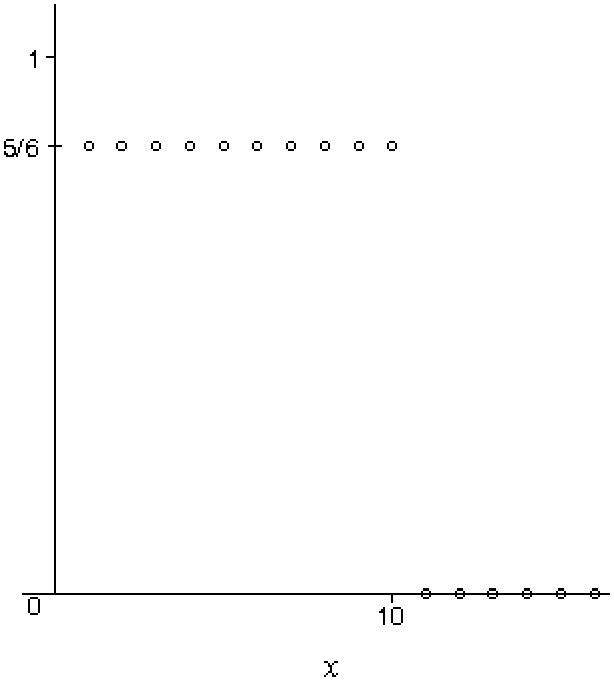

Figure 2. The graph of $\mu_6 \chi_{\{1,2,\ldots,10\}}$

δ_i is a fuzzy topology on X_i. Moreover, if $\nu_\alpha \in \delta_i$ $(\alpha \in \Lambda)$ and $1 = \bigvee_{\alpha \in \Lambda} \nu_\alpha$, then $\nu_\alpha = 1$ for some α. So (X_i, δ_i) is compact.

Now let $(X, \delta) = (\prod_{i \in \mathbb{N}} X_i, \prod_{i \in \mathbb{N}} \delta_i)$. For $(i, n) \in \mathbb{N} \times \mathbb{N}$,

$$\nu_{i,n} = \pi_i^{-1}(\mu_i \chi_{\{1,2,\ldots,n\}}) = \mu_i \chi_{\{1,2,\ldots,n\}} \circ \pi_i \ .$$

is a member of the fuzzy topology $\delta = \prod_{i \in \mathbb{N}} \delta_i$. For a fixed $x = (x_i)_i \in X$,

$$\begin{aligned} \nu_{i,n}(x) &= \mu_i \chi_{\{1,2,\ldots,n\}}(x_i) \\ &= \begin{cases} \frac{i-1}{i} & \text{if} \quad x_i \leq n \\ 0 & \text{if} \quad x_i > n. \end{cases} \end{aligned}$$

Given $\varepsilon > 0$, find i with $1 - \varepsilon < \frac{i-1}{i}$. Then for all $n \geq x_i$, $\nu_{i,n}(x) > 1 - \varepsilon$. So $\bigvee_{(i,n) \in \mathbb{N} \times \mathbb{N}} \nu_{i,n}(x) = 1$. But if S is a finite subset of $\mathbb{N} \times \mathbb{N}$, then we can find $N \in \mathbb{N}$ such that if $(i, n) \in S$ then $n < N$. It follows that for $x = (N, N, N, \ldots)$ we have $\nu_{i,n}(x) = 0$ for all $(i, n) \in S$, and certainly $\bigvee_{(i,n) \in S} \nu_{i,n} < 1$. Thus, we conclude that (X, δ) is not compact.

First Obstacle: A product of arbitrarily many compace fuzzy topological spaces need not be compact.

Another Obstacle: Some constant functions from one fuzzy topological space to another fail to be continuous.

Proposition. *Let (X, δ) be a fuzzy topological space. Then every constant function from (X, δ) into another fuzzy topological space is fuzzy continuous if and only if δ contains all constant fuzzy sets in X.*

Proof. Suppose that every constant function from (X, δ) into any fuzzy topological space is fuzzy continuous, and consider the fuzzy topology γ on $[0, 1]$ defined by $\gamma = \{0, 1, id_{[0,1]}\}$. Let k be a real number, $0 \leq k \leq 1$. The constant function $f : X \to [0, 1]$ with rule $f(x) = k$ for $x \in X$ is fuzzy continuous, and so $f^{-1}(id_{[0,1]}) \in \delta$. But for $x \in X$, $f^{-1}(id_{[0,1]})(x) = id_{[0,1]}(f(x)) = id_{[0,1]}(k) = k$, whence the constant fuzzy set k in X belongs to δ.

Conversely, suppose that δ contains all constant fuzzy sets in X, and consider a constant function $f : (X, \delta) \to (Y, \gamma)$ with rule $f(x) = y_0$. If $\nu \in \gamma$, then for any $x \in X$ we have $f^{-1}(\nu)(x) = \nu(f(x)) = \nu(y_0)$, so that $f^{-1}(\nu)$ is a constant fuzzy set in X and, hence, a member of δ. Thus, f is fuzzy continuous.

Fuzzy Topological Spaces

Part II (May 17)

"Correct" Fuzzification of Topological Spaces: Functors and the General Tychonoff Theorem

In this second part of his presentation, the speaker will discuss Lowen's modified definition of a fuzzy topology on a set and its ramifications for the investigation of fuzzy topological spaces. Emphasis will be placed on the use of category theory as a test for a correct generalization of set-based topology and the success in proving a general theorem on products of compact fuzzy topological spaces.