

# **Optimization in the Undergraduate Curriculum**

**Allen G. Holder**

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**Department of Mathematics  
Rose-Hulman Institute of Technology  
<http://www.rose-hulman.edu/math>**

**Fax (812)-877-8333**

**Phone (812)-877-8193**

# Optimization in the Undergraduate Curriculum

Allen Holder

Department of Mathematics  
Rose-Hulman Institute of Technology  
holder@rose-hulman.edu

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People routinely ponder “how much” or “how little,” and these questions have naturally found their way into the bedrock of mathematics. Indeed, mathematics abounds with the max, min, sup and inf operators, and as such, a study of optimization supports, in some degree, the spectrum of mathematics and its applications. The field’s diversity is one of its greatest strengths, but it is also one of its biggest curricular challenges. For what does it mean to study and teach optimization? You are likely to get different answers from different people, and this article addresses the pedagogical issues of an undergraduate course in optimization. In particular, we position the recent texts *Introduction to Optimization* by Pablo Pedregal [4] and *Understanding and Using Linear Programming* by Jiří Matoušek and Bernd Gärtner [3] within this context.

An undergraduate optimization course differs from the mathematical staples of calculus, linear algebra, analysis and algebra, which are arguably the basis of an undergraduate education in mathematics. Although the organization and delivery of these courses is debated, many of the topics are standardized. Calculus and analysis are built on the concepts of continuity, differentiability and integrability; algebra demands a study of groups, rings and fields; and most linear algebra contains an introduction to matrix algebra and linear transformations. The gateway to optimization is not as well defined. At its core, optimization is the study of problems that can be formulated as

$$\text{opt}\{f(x) : x \in X\}, \tag{1}$$

where  $\text{opt}$  is one of min, max, inf or sup,  $X$  is a set germane to the study, and  $f$  maps  $X$  into some partially ordered set. From this simplistic description we see that students need at least a rudimentary, if not formal, introduction to set theory & functions. Otherwise, the flavor of the course depends on the tools of the students who enter the classroom. The course’s intent can vary along with its level of sophistication, and hence, a course in optimization is available to almost any undergraduate with a few standard prerequisites, such as calculus and linear algebra.

Although there is not a list of standard topics that coalesce to define a stereotypical undergraduate course, the following three themes should be addressed,

**Modeling** The art of modeling is paramount and provides the elemental examples needed to motive insight, rigor, and subtlety.

**Duality and Necessary/Sufficient Conditions** This is optimization's central topic and guides much of the theory to analyze and solve problems.

**Solution Techniques** Optimization has flourished due to the success of efficient algorithms to solve large, meaningful problem classes.

No introductory course can deeply mine all these concepts, but an inauguration into their fundamentals is certainly possible in a first course. It is unnecessary to give equal homage to each, and some courses will naturally focus on one or two. For example, the author's modeling course teaches the process of going from expressed interests and data, to a general model, and then to an analysis of the problem instance defined by the data. In optimization, much of the analysis requires an understanding of duality and the solution procedure, for egregious interpretations are otherwise possible. So, even in a course focused on modeling, a detailed introduction into the other themes is important. A theoretical introduction to duality and/or solution methods is possible without modeling, but this hollows the essence of the discipline. Such a development would be similar to a course in Lebesgue theory without an example of a Lebesgue integrable function that fails to be Riemann integrable. The point is that the beautiful convergence theory supported by Lebesgue theory is appreciated against the backdrop of the Riemann integral's limitations. With good examples, the reason and theoretical restrictions are self-evident. The same is true for optimization. Many problem classes are important because they include meaningful applications, and many of the theoretical nuances are clear with illuminating examples.

A common first course uses linear programming (LP) as the educational vehicle, which is the topic of the text by Matoušek and Gärtner [3]. This type of course is prevalent, and some mistakenly assume that optimization is synonymous with LP (and more generally with mathematical programming). Similarly, others mistakenly interchange optimization with control theory and dynamic programming (controls & DP). Problems in these areas generally combine to form optimization. Both sides are principled on the above themes, and armed with an understanding in one prepares for an understanding in the other. The text of Pedregal [4] succinctly introduces both realms and highlights the parallel themes.

Dividing optimization into mathematical programming versus controls & DP is but one of the many possible splits in a taxonomy. This particular division largely hinges on the nature of the feasible set  $X$ . If we are solving problems over a vector space like  $\mathbb{R}^n$ , then we are within the realm of mathematical programming. If we are instead concerned with a space like the collection of smooth functions, then we are working in controls & DP. Other standard divisions are

<b>Topic</b>	<b>Prerequisites</b>	<b>Audience</b>
Linear Prog.	Calculus, Lin. Alg. (analysis)	2nd - 4th year students. Math, Science, Engineering, Management Sci., Economics
Integer Prog.	Calculus, Lin. Alg. (Computing)	2nd - 4th year students. Math, Computer Science, Management Science
Opt. Modeling	Calculus, Lin. Alg. (Diff. Eq., Prob. & Stat., Computing)	2nd - 4th year students. Math, Science, Engineering, Management Sci., Economics
Comb. Opt.	Graph Theory, Lin. Alg.	2nd - 4th year students. Math, Computer Science, Management Science
Nonlinear Prog.	Analysis, Lin. Alg.	3rd - 4th year students. Math, Engineering
Control Theory	Analysis, Lin. Alg.	3rd - 4th year students. Math, Engineering
Mathematical Opt.	Analysis, Line. Alg. (algebra, graph theory)	3rd - 4th year students Math

Table 1: Some example topics for a first course. Parenthetical prerequisites are suggested depending on the level of desired rigor.

deterministic versus stochastic, linear versus nonlinear, and continuous versus discrete. A first course should live within one of the main subdivisions. As an example, it is likely overly ambitious to introduce both deterministic and stochastic variants of nonlinear programming within a standard semester/quarter while at the same time sufficiently introducing the above themes.

Characteristics for a few gateway courses are listed in Table 1, which is but a whisper of the possibilities. The purposeful omission of operations research (OR) deserves comment. While OR and optimization have a significant overlap, OR includes additional topics that limit an introduction into optimization's central themes within the confines of a single course. For example, OR would typically include discussions on queuing theory and simulation, which play a role in optimization but are disciplines within themselves. OR is innately an applied discipline that brings to bear whatever mathematical tools aid a problem's solution, and its study is consequently spread over topics in and out of optimization.

The prerequisites in Table 1 indicate the topics needed to succeed and/or a comparable level of mathematical maturity. Calculus is listed for linear and integer programming, but it is possible to introduce the needed concepts within the most basic of courses. Most curricula require some calculus before linear algebra, making the calculus prerequisite tacit. Those listed with an analysis prerequisite indicate that the theoretical development is rigorously similar to a standard first course in analysis or advanced calculus. The prerequisites can

vary depending on the intent of the course. If stochastic problems are to be considered, then a prerequisite in probability is appropriate. The topics in Table 1 are intentionally deterministic since the stochastic counterparts have an increased level of sophistication that complicates the fundamental theory. If probability theory is standard among the student population, a stochastic version of any of these is possible, if not desirable.

From the author's experience, students generally enter a course with two weaknesses. First, they are able to mechanically perform multivariate calculus but lack the command and geometric insight to use this material in a theoretical framework. This sentiment is not atypical and is the impetus for *Div, Grad, Curl and all That* [5]. Second (and similarly), students have mechanical skills from linear algebra but little control of the associated theory. These shortcomings lengthen the time required to present simple concepts, like the definition of a hyperplane, gradient descent, or analytic equivalents of convexity. This is not a deriding reflection on the calculus and linear algebra we all teach; rather, the fact is that most students need time to wield these concepts. The subtlety of mathematics is striking, and we should not forget that several great minds made mistakes. Unlike the author, who initially overestimated student abilities, first time instructors should be cautious about the course's pace since succinct reviews are warranted. As discussed momentarily, combining material from several courses is one of optimizations strengths within the undergraduate curriculum.

Some (graduate) curricula sequence their optimization courses, with LP being the conventional introduction. This makes sense if there is an ample menu of continuations, but undergraduate courses should be tailored to the curricular goals of the environment and to the flexibility of the program. An undergraduate degree only introduces the mathematical discipline, and providing a window into the many realms of mathematics is not possible. By the time a student completes the staples mentioned earlier together with other, more common courses such as differential equations, number theory, combinatorics, geometry, topology, etc..., there are only a few opportunities to broaden and/or deepen the undergraduate experience. In such a situation, what is the reason to offer a course in optimization? Two answers come to mind.

- Optimization builds on several mathematical staples and fills the pedagogical role of a capstone course that re-emphasizes and solidifies previous coursework.
- Optimization is a natural conduit to other disciplines.

The first of these is directed at a course in the later part of an undergraduate curriculum. Such a course could combine topics from analysis, algebra, linear algebra, and combinatorics. Few students would be able to acquire each of these as a prerequisite, but as long as a student enters the course with an understanding of and the maturity from a couple of these courses, topics in the others can be introduced to promote optimization. This type of course broadens and affirms the undergraduate repertoire and provides a rigorous foundation for

further study in optimization. The second reason highlights the fact that optimization is well positioned to support the interdisciplinary nature of modern mathematics. From the author's experience, the trend among undergraduate mathematics majors is to have a second major. In such an environment, an optimization course can support the educational hand-off between the disciplines. This intermediary role is supported by abundant examples in science and engineering, but also extends to art, see the work of Bosch [1], and the social sciences, especially within economics. Moreover, there is the flexibility to include a quantitative/computer science component, which is sometimes desired. So, optimization is a handy curricular tool that broadens departmental offerings and advances other educational directives. Also, the level of rigor and application can be tuned to meet curricular desires.

Although their objectives are different, the two texts of this article amply meet the needs of a first course in optimization and are quality classroom companions. Neither is written expressly for students of mathematics, but both have a pleasant mix of application and theory. The level of rigor approaches a third or fourth year student who has already had calculus, linear algebra, and possibly differential equations.

In the preface of *Introduction to Optimization*, Dr. Pedregal indicates the intended audience is science and engineering students. The text's goal is to show that "mathematical programming, variational problems, and optimal control problems are explained and integrated as a unity." The author beautifully accomplishes this objective in a succinct 233 pages, including homework problems and a long, well-explained collection of meaningful examples. The examples initiate the presentation and are re-visited as the material to analyze them is developed. The author immediately and successfully establishes that the art of modeling is an essential part of the discipline. Students in physics, engineering and applied mathematics will warm to this beginning and will be encouraged to learn the mathematics that follows.

The rest of the text is staged with a discussion of mathematical programming, with chapters on LP (including the simplex algorithm), nonlinear programming, and nonlinear solution techniques, followed by chapters on variational problems & DP and control theory. The strength of this progression is that it highlights the role of the above mentioned themes across the discipline. The theoretical content is presented in a 'conversational' form and reminds the author of the style used by Churchill and Brown in *Complex Variables and Applications* [2]. Several concepts are illustrated with example instead of proof, but the central theory is clear and well established. The use of examples to flirt with important concepts just beyond the central themes is expected and necessitated by the author's goal to present the "unity" that encompasses the entirety of the field.

A limitation of the presentation is that the mathematical results are stated in a simple form, which reduces subtlety and rigor. For example, the discussion on LP routinely refers to **the** optimal solution and claims that a unique optimal solution "is the most desirable situation." It is certainly desirable from a presentation perspective since it streamlines development, but the most desirable

situation is the one that most accurately describes the phenomena and that is mathematically sound. This particular issue presents itself with the common misconception that **the** dual solution provides marginal information, which is generally false in the presence of degeneracy. Nothing in this discussion is technically incorrect or purposefully misleading since the author's goal is to give insight instead of rigor. The development of the Karush-Kuhn-Tucker conditions similarly assumes the simplifying condition of regularity and states that singularities are "beyond the scope of this text," which is true and exactly the point. If these and related issues were developed, the text would lose its lean attractiveness of introducing the preeminent topics over the entirety of the field.

The reliance on intuition and example increases as the text continues with DP and controls in Chapters 5 and 6, where Bellman's equation and Pontryagin's maximum principle are presented. The examples highlight the results and hint at the underlying mathematics, and importantly, they give the insight needed to solve problems. On this later point, the author includes a brief discussion about discretizing continuous problems for numerical approximation. Again, the fact that these results are intuitively motivated is necessitated by the overriding pedagogical objective. Just as it would be daunting, if not impossible, to begin a course with separable differential equations and conclude with a rigorous presentation of Krylov spaces, it is unrealistic to expect a mathematically robust presentation of controls & DP. The point is that the author has found a way to start with the fundamentals of mathematical programming and end with an intuitive glimpse into the mathematics of controls & DP.

A mathematics course based on this text is well suited to students of mathematics and engineering. In a single course, students see the mathematical pillars of a theory driven by rich and meaningful examples and leave with a broad understanding of optimization as an applied mathematical discipline. They will also be prepared to advance their understanding of the sub-disciplines. Few texts introduce the desired unity of the discipline, and none to the authors' knowledge do so as succinctly and as cleanly. Augmented with theoretical developments, especially for the material on controls & DP, this text could span two courses.

The text by Matoušek and Gärtner [3] has the different target audience of computer science students, with their guiding phrase being "what every theoretical computer scientist should know about linear programming." This text almost exclusively addresses LP and is thus similar to many LP texts that are used for an introductory course. The text begins with a geometric description and a brief collection of traditional examples like the diet problem, the problem of finding a separating hyperplane, the problem of maximizing flow in a network, and 1-norm regression. As with the other text, the idea is to use modeling as motivation. More interesting applications await the reader at the end of the text.

The book continues with a look at integer programming, which focuses on the three problems of finding 1) a maximum weight matching, 2) a minimum vertex cover, and 3) a maximum independent set. These are classic problems and are purposefully selected to highlight the varied difficulties associated with

integer programs. The first problem is shown to be solvable by relaxing the integral constraints. The second is known to be NP-hard, but the authors use the relaxation to develop a heuristic that renders a solution with an objective value no worse than twice the best possible. The integral relaxation for the third is shown to provide no information about the best objective value under the integral constraint, and hence, the scheme used in the first two examples fails. Introducing these concepts early in the text is not standard but is appropriate for the target audience since these problems are often considered by computer scientists. Moreover, this material is not an introduction to integer programming but is rather a good description of how the continuous relaxation can or can not approximate the associated integer problem. This presentation assumes a basic knowledge of complexity, a topic that is not likely to be understood by mathematics students.

LP theory and solution methods are alternately discussed in the next four chapters. The authors do not take the shortest route through this material and thankfully discusses convexity and polyhedral theory, discussions that would not have been needed if the goal had been to only develop the simplex method and the strong duality theorem. This is one of the book's strengths, as the authors seem intent on educating the reader about how LP's theory flows from more general and powerful results. The lack of a similar intent is the downfall of some texts used in the mathematics classroom. If the goal is to provide a first course in optimization from the LP perspective, then LP should be used to illuminate results within the field and not solely be viewed as a particularly special problem class with special proofs. Matoušek and Gärtner balance their presentation between proofs based on the linearity and those that do not. As an example, the text provides two proofs of the duality theorem, one based on the simplex algorithm, which is specific to LP, and another on Farkas' lemma, which is a more general result. Three separate derivations of Farkas' lemma are included, a traditional argument based on separating hyperplanes, another on minimally (irreducible) infeasible subsystems (a proof that was new to me), and a final argument that uses Fourier-Motzkin elimination. This is a wonderful development for mathematics students since each proof is in its own right a little gem of thought and creativeness.

Solution methods are not limited to the simplex method, and the basics of an ellipsoid method and an interior point method are present. The discussion of the simplex method is robust and includes different pivot schemes and a proof that cycling is not possible under Bland's rule. It also hints at the fact that an LP algorithm is searching for a partition of the variable indices, a point often missed. The other techniques lack the same level of rigor, with the only complete result being that the central path is well defined under a rank condition. A benefit of this material is that it requires Newton's method, which again, places LP within the broader context of optimization. Newton's method is not specifically mentioned, but the technique is evident.

The final chapter is titled "More Applications," but these are not blaisé models like those mentioned earlier in the text. Rather, this is a collection of theoretical applications, and the authors impressively demonstrate how the

theory of LP can be used as a tool to prove other results. Some of the areas considered are zero sum games, coding theory, and the search for sparse solutions (those with small support sets). This collection is a welcome addition to an introductory course since it begins to show how optimization's tentacles reach into other disciplines. The text concludes with a substantive glossary that extends beyond the covered topics.

Although there are numerous favorable qualities, the role of [3] within a standard mathematics classroom is limited for three reasons. First, there are no homework problems. Second, as with the first text, there are places where proofs are either sketched or simply ignored. The authors place many of the proofs in indented text and state that "Whoever finds these passages incomprehensible may freely ignore them." This mentality may be fine in other areas, but mathematicians are all about the proofs. Third, as mentioned above, there are areas where some knowledge of computer science is assumed, and this material will likely need supplementation. If the instructor is willing to augment the text, then this would be an exceptional basis for a mathematics course that bridges computer science. It would certainly be a superb auxiliary text for an undergraduate course on LP.

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