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Abstract

Consider a surface, S, with a *kaleidoscopic* tiling by non-obtuse triangles (tiles), i.e., each local reflection in a side of a triangle extends to an isometry of the surface, preserving the tiling. The tiling is *geodesic* if the side of each triangle extends to a closed geodesic on the surface consisting of edges of tiles. The reflection group G^*, generated by these reflections, is called the *tiling group* of the surface. This paper classifies, up to isometry, all geodesic, kaleidoscopic tilings by triangles, of hyperbolic surfaces of genus up to 13. As a part of this classification the tiling groups G^* are also classified, up to isometric equivalence. The computer algebra system Magma is used extensively.

1 Introduction

Let S be an orientable, compact surface of genus σ. A *tiling* $\mathcal{T}$ of S is a complete covering of the surface by non-overlapping polygons, called *tiles*. An example of such a tiling is the icosahedral tiling of the sphere $S = \mathbb{S}^2$ given in Figure 1.1. In this example the tiles are triangles, and we shall assume that this is the case for all tilings throughout the paper. Observe that in this tiling, the reflection in the great circle determined by an edge of a tile isometrically maps tiles to tiles. Moreover, the great circle fixed by one of these reflections - called the *mirror of the reflection* - is a union of edges of tiles. We call a tiling satisfying these two properties a *kaleidoscopic, geodesic* tiling. A precise definition will be given later. Now, the reflections in the edges of the tiles generate a group G^* of S preserving the tiling, which we shall call the *tiling group* generated by $\mathcal{T}$. For the icosahedron this group is isomorphic to $\mathbb{Z}_2 \times A_5$, a group of order 120, exactly the number of tiles on the surface. In this paper we shall classify all triangular tiling groups – and hence all triangular tilings – up to isometry on surfaces of genus 2 through 13.

As a by-product of our classification we shall also classify the triangular groups of automorphisms of surfaces of genus 2 through 13. Let us digress briefly

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to understand the terms and the significance. Observe that G^* has a subgroup G of index 2 consisting of the orientation-preserving automorphisms of S. For the icosahedral tiling G is isomorphic to A_5. The key feature of G is that the orbit space S/G is a sphere and the projection $\pi : S \to S/G$ is branched over three points corresponding to the three types of vertices in the tiling.

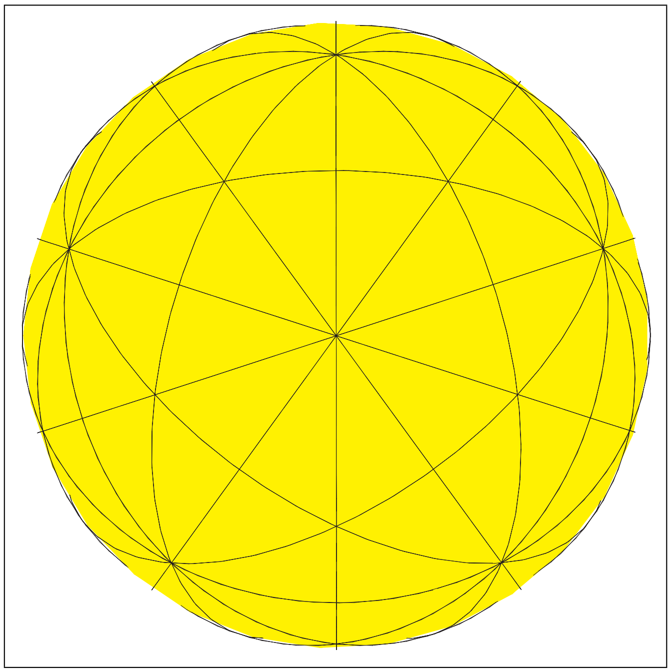

Fig 1.1. Icosahedral tiling - top view

To answer our question on classification of the tiling groups and tilings we will also answer two seemingly more ambitious questions.

Problem 1. For a surface S with genus σ, satisfying $2 \leq \sigma \leq 13$, determine, up to isometry, all groups H of conformal isometries such that S/H is a sphere and such that $S \to S/H$ is branched over three points. These will be called *triangular rotation groups* as explained later.

Problem 2. If H is as above, then is H the subgroup G of orientation-preserving automorphisms of the tiling group G^* of some tiling of the surface S? Classify the groups G^*, so obtained, up to isometry.

We are concentrating on triangular tiling groups because

- such groups are large in comparison to the genus of the surface, and thus there is a strong link between the geometry of the surface and the structure of the group,
- the group theory is interesting,

- there are no moduli to deal with, (see below)
- up to topological equivalence, almost all finite group actions on a surface arise from subgroups of triangle tilings.

In [2], Broughton classifies all possible groups of orientation-preserving automorphisms of surfaces of genus 2 and 3. Thus the present paper is a partial extension of this work to higher genus in addition to answering the additional question on tilings. The classification of orientation-preserving group of surfaces has been, investigated in genus 4 and 5 under various equivalence relations (see the references and discussion in [2]). The present work utilizes the work of Broughton [2] in genus 2 and 3, the work of Vinroot [12] in genus 4 and 5, the work of Dirks and Sloughter [4] in genus 6 and 7, and some additional calculations and refinements by Broughton.

1.1 Connection to Fuchsian groups and moduli spaces.

All conformal equivalence classes of surfaces of a given genus $\sigma \geq 2$ may be parametrized by a quasi-projective variety $\mathfrak{M}_\sigma$, of dimension $3\sigma - 3$, called the moduli space. Indeed, every surface S of genus σ may be represented as a quotient $\mathbb{H}/\Gamma$ where $\mathbb{H}$ is the hyperbolic plane (disc model) and Γ is a fixed point free Fuchsian group of automorphisms with the following presentation

$$\Gamma = \langle \alpha_1, \ldots, \alpha_\sigma, \beta_1, \ldots, \beta_\sigma : [\alpha_1, \beta_1] \cdots [\alpha_\sigma, \beta_\sigma] = 1 \rangle .$$

Now $\mathcal{L} = \left\{ \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} \in GL_2(\mathbb{C}) : a\overline{a} - b\overline{b} = 1 \right\} / \{\pm I\}$ is the group of automorphisms of $\mathbb{H}$, acting by linear fractional transformations. The group Γ is unique up to conjugacy in $\mathcal{L}$. The moduli space may be represented as the following orbit space

$$\mathfrak{M}_\sigma = \{(a_1, \ldots, a_\sigma, b_1, \ldots, b_\sigma) \in \mathcal{L}^{2\sigma} : [a_1, b_1] \cdots [a_\sigma, b_\sigma] = I\}^o / \mathcal{L}$$

where the superscript ^{o} indicates the open subset of 2σ-tuples generating discrete subgroups. In this setup the conformal automorphism group $\mathrm{Aut}(S)$ of the surface $S = \mathbb{H}/\Gamma$ is $N_{\mathcal{L}}(\Gamma)/\Gamma$ and there is an exact sequence

$$\Gamma \hookrightarrow N_{\mathcal{L}}(\Gamma) \twoheadrightarrow \mathrm{Aut}(S). \tag{1}$$

Two surfaces S and S' with conformal automorphism groups $\mathrm{Aut}(S)$ and $\mathrm{Aut}(S')$ are said to be equisymmetric if there is an orientation-preserving homeomorphism $h : S \to S'$ that induces an isomorphism $\mathrm{Ad}_h : \mathrm{Aut}(S) \to \mathrm{Aut}(S')$, $g \to h \circ g \circ h^{-1}$. As discussed and proven in [3], the set of all conformal equivalence classes of surfaces, equisymmetric to a given surface, form a smooth, connected, locally closed subvariety of $\mathfrak{M}_\sigma$, called an equisymmetric stratum. The complex dimension of the stratum is $3\rho - 3 + t$, where the quotient surface $S/\mathrm{Aut}(S)$ has genus ρ and the quotient map $S \to S/\mathrm{Aut}(S)$ is branched over t points. The surfaces under study in this paper simply correspond to 0-dimensional strata, or simply points, except as noted following.

Suppose that G is any group of automorphisms of S. Let Λ be the subgroup of $N_{\mathcal{L}}(\Gamma)$ that maps onto G in equation 1. Then we also have an exact sequence

$$\Gamma \hookrightarrow \Lambda \twoheadrightarrow G.$$

Sequences of this type give a 1-1 correspondence between the subgroups G of $\mathrm{Aut}(S)$ and the intermediate subgroups $\Gamma \trianglelefteq \Lambda \subset N_{\mathcal{L}}(\Gamma)$. Define for Λ, or indeed for any for any Fuchsian group, $d(\Lambda) = d(G) = 3\rho - 3 + t$ where ρ and t are defined as above except with respect to the projection $\mathbb{H} \longrightarrow \mathbb{H}/\Lambda$. Suppose that $d(\Lambda) > d(N_{\mathcal{L}}(\Gamma))$. Then there will a different surface $S' = \mathbb{H}/\Gamma'$ and a Λ' isomorphic to Λ – topologically conjugate, but not conjugate in $\mathcal{L}$ – such that $d(\Lambda) = d(\Lambda') = d(N_{\mathcal{L}}(\Gamma'))$. Thus we may deform S to a surface S' with a smaller automorphism group, isomorphic to G (except in a small number of cases, see below). In $\mathfrak{M}_\sigma$, the closure of the equisymmetric stratum of S' will contain the stratum of S. Thus, in some sense the group G defines a stratum of $\mathfrak{M}_\sigma$. Now there is the possibility that $d(\Lambda) = d(N_{\mathcal{L}}(\Gamma))$. In this case the equisymmetric strata determined for G and $\mathrm{Aut}(S)$ coincide. In [11] Singerman classifies the set of all possible pairs of distinct Fuchsian groups $\Lambda_1 \subset \Lambda_2$ for which $d(\Lambda_1) = d(\Lambda_2)$ (we always have $d(\Lambda_1) \geq d(\Lambda_2)$). Singerman showed that $d(\Lambda_i) = 0$ or 1 and that there are a finite number of infinite families of pairs and a finite number of exceptional cases. Thus, in order to apply the results of Problem 1 to the classification of the zero dimensional strata of the moduli space, the additional work of determining when $G = \mathrm{Aut}(S)$ is required, since we would need to remove those G with strict containment $G \subset \mathrm{Aut}(S)$. That classification is dependent on the genus, and is still incomplete, so its inclusion would take us beyond the scope of this paper.

A real surface S is one which has some representation as a solution of a set of complex equations with real coefficients in some complex projective space. The complex conjugation map defines an anti-conformal involution on S. More generally any anti-conformal involution on a surface is called a *complex conjugation* or *symmetry.* It is well known that a surface is symmetric, i.e., has a symmetry, if and only if it is real. Thus, the answer to Question 2 will provide information on classifying those 0 dimensional strata corresponding to real curves, for in case of a maximal triangular group, a complex conjugation must have a fixed point set passing through two neighbouring vertices and in this case it must be a reflection in the side of some triangle. It follows that the entire tiling is kaleidoscopic.

Remark 1 *Though it is traditional to phrase and solve problems on surfaces through the mechanism of Fuchsian groups we will adopt a strategy of considering tiling group G^* and its subgroup G as the object of primary interest, since all the computations are phrased in terms of finite groups. However, on occasion we will refer to the Fuchsian group construction above to prove results and to transfer theorems about tilings to classification of surfaces in the moduli space.*

1.2 Outline

The outline of the paper is as follows. In section 2 we define the tilings we are interested in and introduce the tiling group. We may thereby transform the question of classification of tilings into one of determining finite groups generated in certain specific ways, up to isomorphism. The groups in question satisfy $|G| \leq 84(\sigma - 1) \leq 1008$. Since Magma (also GAP) has a data base of groups of order less than 1000, Question 1 can be answered for genera in the given range by computation with Magma and a little cleverness (mostly hidden in the construction of the database). Question 2, on the extension of a triangular rotation group to a tiling group, can be translated into the existence of an involutary automorphism θ with certain properties. In section 3 we give an overview of how to organize the calculation. We also give a simple implementation in Magma of how to find θ and other automorphisms. It turns out that a brute force construction works well here for small group orders. Even though we have powerful computer tools such as Magma at our disposal, and a database, some thought is required to make efficient use of it. Therefore in section 4 we discuss some general methods that allow us to restrict the structure of groups and effectively apply Magma. The tables we present (sections 6 and 7) show that there are very few rotation group that do not come from tiling groups; 41 out of 513 cases. By way of contrast, in section 5 we construct an infinite class of rotation groups which cannot be tiling groups. Finally in sections 6 and 7 we present the classification. Since there are 513 cases, we only give detailed tables for genus 2 to 7, in section 6, the scope of the previous works mentioned. In section 7 we give summary information tables for the entire classification, in addition to a less detailed listing of all rotation and tiling groups. Complete electronic databases of the actions are available at the website [17].

Computer computations All large group theoretic calculations were performed using the Magma software package [15], though they could also be easily implemented in the GAP package [14]. Some other calculations were done using the Maple package [16]. All scripts used to generate the tables are available at the website [17].

Acknowledgments The preliminary investigations [12] and [4] were completed at the Rose-Hulman Institute of Technology NSF-REU program, during the summers of 1997 and 1998 respectively, under the guidance of the first author. The authors of those papers thank the NSF for support for their research under grant #DMS-9619714.

2 Tilings and tiling groups

Let S and $\mathcal{T}$ be a surface and tiling as described in the introduction. The sides or edges of the tiles are called the *edges of the tiling* and the vertices of the tiles are called the *vertices of the tiling.* Let $\mathcal{E}$ and $\mathcal{V}$ denote the collection of edges and vertices of the tiling. We assume that the surface has some Riemannian metric of constant curvature, and that all geometric notions such as polygon, edge, straight line segment or reflection are with respect to this metric. We are primarily interested in the case of a hyperbolic geometry where the genus $\sigma \geq 2$, and the curvature is negative.

Definition 2 *A tiling $\mathcal{T}$ of a surface said to be a* kaleidoscopic tiling *if the following condition is met.*

- *For each edge $e \in \mathcal{E}$ of the tiling the local reflection r_e in the edge e is an isometry of the surface that maps tiles to tiles. In particular it interchanges the two tiles whose common edge is e.*

A tiling $\mathcal{T}$ is called a geodesic, kaleidoscopic tiling if the following additional condition is met.

- *The* fixed point subset *or* mirror $S_{r_e} = \{x \in S : r_e(x) = x\}$ *of each reflection r_e, is the union of edges of the tiling.*

Remark 3 *The fixed point subset of any orientation-reversing isometry is automatically a disjoint union of circles called ovals. Each oval is locally geodesic.*

Remark 4 *The dodecahedral tiling of the sphere by twelve pentagons is an example of a kaleidoscopic tiling which is not geodesic. There are many other examples in higher genus.*

2.1 The tiling group

The reflections in the edges of a tiling generate a group of isometries of the tiling, called the *tiling group.* We describe this group in some detail now for the case of triangle. The generalization to the group of a general polygon easily follows from the triangle discussion. It is easy to show that every tile in the plane is the image, by some element of the tiling group, of a single tile, called the *master tile,* pictured in Figure 2.1. The sides of the master tile, Δ_0, are labeled p, q, and r, and we denote the points opposite these sides by P, Q, and R, respectively. We also denote by p, q, and r the reflection in corresponding side. Now, because of the geodesic condition, there is an even number of equal angles at each vertex. Thus the angles at P, Q, R have size $\frac{\pi}{l}$ radians, $\frac{\pi}{m}$ radians, and $\frac{\pi}{n}$ radians, respectively, where l, m, and n are integers ≥ 2. We say that Δ_0 is an (l, m, n)-triangle (see Figure 2.1). At each of the vertices of the triangle, the product of the two reflections in the sides of the triangle meeting at the vertex is a rotation fixing the vertex. The angle of rotation is twice the angle at this vertex. For example the product $a = pq$ - a reflection first through q then through p is a counter-clockwise rotation through $\frac{2\pi}{l}$ radians. Rotations around each of the other corners can be defined in the same way, so that $b = qr$ and $c = rp$ are counterclockwise rotations through $\frac{2\pi}{m}$ radians and $\frac{2\pi}{n}$ radians, respectively.

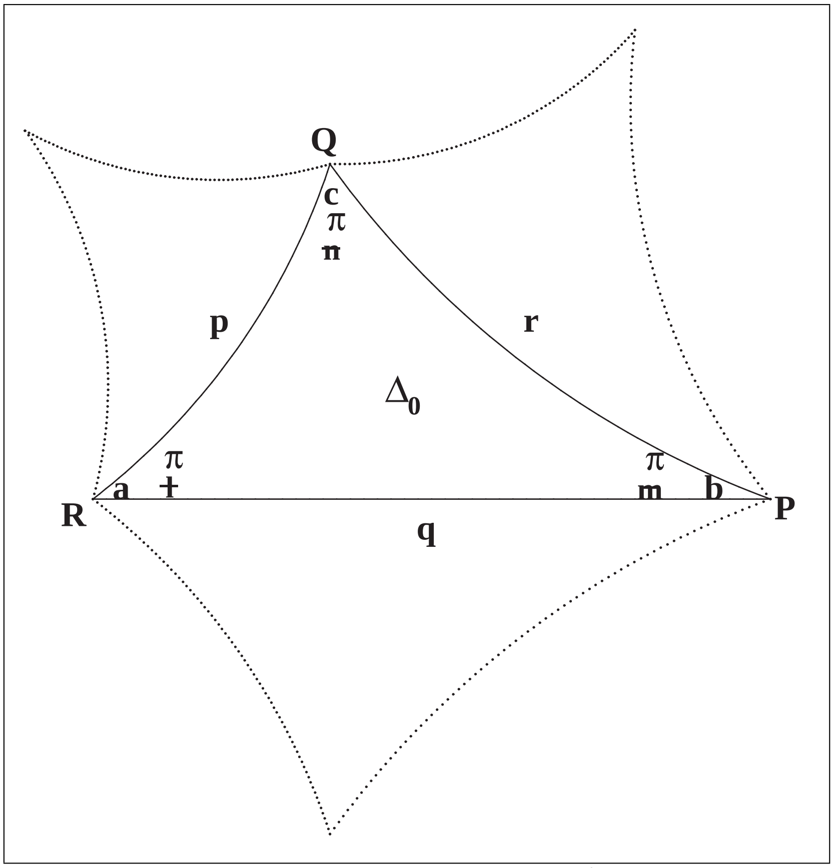

Fig 2.1. The master tile and generators of T^* and T

From the geometry of the master tile, we can derive relations among these group elements. It is clear that since p, q, and r are reflections, the order of each of these elements is 2:

$$p^2 = q^2 = r^2 = 1. \tag{2}$$

From the observations about rotations above, it is also clear that

$$a^l = b^m = c^n = 1, \tag{3}$$

or more exactly

$$o(a) = l,\ o(b) = m,\ o(c) = n, \tag{4}$$

and

$$abc = pqqrrp = 1. \tag{5}$$

Let $G^* = \langle p, q, r\rangle$ and $G = \langle a, b, c\rangle = \langle a, b\rangle$ be the groups generated by the above elements. The subgroup G is the subgroup of orientation-preserving isometries in G^*. This subgroup a normal subgroup of G^* of index 2. In fact $G^* = \langle q\rangle \ltimes G$, a semi-direct product.

Definition 5 *The group G^* defined above will be called the* full tiling group *or just the* tiling group *of S (with its given tiling). The subgroup G will be called the* orientation-preserving tiling group *or OP tiling group.*

Remark 6 *There may be isometries of S that preserve the tiling but are not contained in G^*. In this case the full symmetry group of the tiling has a factorization $U \ltimes G^*$, where U is the stabilizer of the master tile. This may be proven by noting that G^* acts simply transitively on the tiles of S as a consequence of the Poincaré Polygon theorem. Since U is a group of isometries of a tile it is a subgroup of Σ_3, the symmetric group on the vertices of the master tile. Since the term "symmetry group" of an object usually refers to the totality of self-transformations of the object preserving the given structure, the use of the term symmetry group to describe G^* would be misleading. Therefore, we use the term tiling group.*

Remark 7 *Most of the statements above and to follow can be proven by working in the universal cover $\mathbb{H}$ and working with the covering Fuchsian groups of G and G^*. See the subsection entitled Connection to Fuchsian groups at the end of this section.*

Rotation groups The group G is generated by rotations at the vertices of the tiling $\mathcal{T}$, in fact by rotations a, b and c at the three points P, Q and R. We will call a group of conformal isometries of a surface generated by rotations at points on the surface a *rotation group.* This is analogous to the term reflection group. It is easily shown that for the rotation group G the quotient S/G is a sphere obtained by sewing together two adjacent triangles. Furthermore the projection is $\pi : S \to S/G$ is branched over exactly three points of orders l, m and n. More precisely the stabilizer $G_R = \{g \in G : gR = R\} = \langle a\rangle$, $G_P = \langle b\rangle$, $G_Q = \langle c\rangle$ and G acts freely on all other points except those G-equivalent to P, Q, or R. Call any group G, whose action on S satisfies these two conditions, whether it is an OP tiling group or not, a *triangular rotation group.* It is easily shown that every triangular rotation group has a triple of elements (a, b, c) which generates G and satisfies (4) and (5). Such a triple is a called a *generating* (l, m, n)*-triple* of G. The elements a, b and c can be chosen as generators of stabilizers of a triple of points on S.

Remark 8 *Sometimes we need to distinguish isomorphic but possibly distinct rotation groups of surfaces. In this case we will say that an arbitrary finite group acts as a rotation group on S if there is a monomorphism $\varepsilon : G \to \mathrm{Aut}(S)$ such that the image $\varepsilon(G)$ is a triangular rotation group as described above. Similar terminology applies to a group G^* acting as a tiling group. In this case we need to replace* $\mathrm{Aut}(S)$ *by the group of isometries* $\mathrm{Aut}^*(S)$ *of S.*

In the first major work on automorphisms of surfaces by Hurwitz [7], we find the following formula (specialized for our purposes), known as the Riemann-Hurwitz equation, relating the genus of the surface, the order of the rotation group, and the orders of the generating rotations:

$$\frac{2\sigma - 2}{|G|} = 1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right) \tag{6}$$

It follows that the genus is given by:

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right), \tag{7}$$

and the group order by:

$$|G| = \frac{2\sigma - 2}{1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)}. \tag{8}$$

We see that a tiling gives a group, what about the reverse direction? First we shall see that triples give triangular rotation groups. The theorem follows from the Riemann existence theorem, or standard arguments with Fuchsian triangle groups.

Theorem 9 *Let G have a generating (l, m, n)-triple. Then there is a surface S of genus σ with an orientation-preserving G-action such that G is a triangular rotation group on S.*

Remark 10 *A proof using Fuchsian groups shows that σ is automatically an integer, though it can be shown algebraically that the existence of a generating triple guarantees that σ in an integer.*

Example 11 *Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. The group G has 96 generating $(4, 4, 4)$-triples and the value of σ given by equation 7 is 3. Thus there is a potential for 96 different types of rotation groups isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$ on surfaces of genus 3. We shall see presently that all these rotation groups come from tilings on the surfaces, and that the tilings are all isometrically equivalent.*

Extending rotation groups to tiling groups The conjugation action of q on the generators a, b of G induces an automorphism θ satisfying:

$$\begin{aligned} \theta(a) &= qaq^{-1} = qpqq = qp = a^{-1}, \\ \theta(b) &= qbq^{-1} = qqrq = rq = b^{-1}. \end{aligned} \tag{9}$$

An automorphism satisfying the equations above is unique if it exists, since it is specified on a generating set. The automorphism θ is involutary, $\theta^2 = id$, and non-trivial if one of l or m is greater than 2. The following theorem allows us to extend the action of a rotation group to a tiling group, see [10].

Theorem 12 *Let G be a triangular rotation group on a surface S with a generating (l,m,n)-triple (a,b,c). If, in addition, there is an involutary ($\theta^2 = id$) automorphism θ of G satisfying (9) then surface S has a tiling $\mathcal{T}$ by (l,m,n)-triangles such that OP tiling group as constructed above is the original G, the triple (a,b,c) is constructed from the master tile as above, and such that $G^* \simeq \langle\theta\rangle \ltimes G$.*

Remark 13 *A G-invariant tiling always exists on the surface, obtained by projecting a tiling on the universal cover. However, none of the local reflections extend to the entire surface, unless the tiling is kaleidoscopic..*

Example 14 *Continuation of Example 11: Since G is abelian in this case the inversion map $\theta : (x,y) \to (-x,-y)$ is an automorphism satisfying 9. Thus, there is at least one tiling of a surface of genus 3 by 32, $(4,4,4)$- triangles. This example shows that an abelian rotation group is always an OP tiling group via the automorphism $\theta(g) = g^{-1}$ for all $g \in G$.*

2.2 Isometric equivalence

Example 14 prompts the following question. How many of the 96 different triples yield different tilings. This is answered by the following definition and theorem.

Definition 15 *Suppose that G acts as a triangular rotation group on two surfaces S, S' defined by maps $\varepsilon : G \to \mathrm{Aut}(S)$ and $\varepsilon' : G \to \mathrm{Aut}(S')$. Then we say that the actions are* isometrically equivalent *if there is an isometry $h : S \to S'$ and an automorphism ω of G^* such that*

$$h(\varepsilon(\omega(g)) \cdot x) = \varepsilon'(g) \cdot h(x) \text{ for all } x \in S, \tag{10}$$

i.e.,

$$\varepsilon' = Ad_h \circ \varepsilon \circ \omega$$

where $Ad_h : \mathrm{Aut}(S) \to \mathrm{Aut}(S')$, $g \to hgh^{-1}$. If in addition S and S' have assigned orientations and h is orientation-preserving then h is said to be a conformal equivalence *and that the actions of G are* conformally equivalent. *Similar definitions hold for the action of a tiling group G^*, except that isometrically equivalent actions are automatically conformally equivalent. In the simplest case where $S = S'$ and $h = id$, and the generating triples (a,b,c) (a',b',c') and automorphisms θ and θ' are related by $(a',b',c') = \omega \cdot (a,b,c)$ and $\theta' = \omega\theta\omega^{-1}$.*

Remark 16 *An isometry which is not conformal preserves the measure of angles but reverses their sense. Therefore, it is called anti-conformal. An isometric equivalence of G-actions automatically induces an equivalence of G^*-actions provided that one of the G-actions extends to a G^*-action. In turn, a conformal equivalence of tiling group actions automatically is an isometric equivalence of tilings.*

The proceeding remark says that the classification of tiling groups and tilings is a direct consequence of the classification of rotation groups. The two theorems following give a precise statement on when two rotation groups are equivalent.

Theorem 17 *Let G be a triangular rotation group on a surface S with a generating (l,m,n)-triple (a,b,c). Let S' be another surface of the same genus with G-action defined by an (l,m,n)-triple (a',b',c'). (If the genus of S is 1, then assume that both surfaces have the same area.) Then there is a conformal equivalence $h : S \to S'$ commuting with the group actions of G if and only if one of the following holds:*

1. *l, m, n are distinct and there is an $\omega \in \mathrm{Aut}(G)$ such that*
$$(a', b', c') = \omega \cdot (a, b, c),$$
holds;

2. *$l = m \neq n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (b, a, a^{-1}ca)$$
holds;

3. *$l \neq m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (b^{-1}ab, c, b)$$
holds;

4. *$l = n \neq m$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (c, c^{-1}bc, a)$$
holds; and finally

5. *$l = m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a, b, c),$$
$$(a', b', c') = \omega \cdot (b, a, a^{-1}ca),$$
$$(a', b', c') = \omega \cdot (b^{-1}ab, c, b),$$
$$(a', b', c') = \omega \cdot (c, c^{-1}bc, a),$$
$$(a', b', c') = \omega \cdot (b, c, a),$$
$$(a', b', c') = \omega \cdot (c, a, b).$$
holds.

Remark 18 *If $S = S'$ then $(a', b', c') = \omega \cdot (a, b, c)$ is the only equivalence above that preserves the type of vertices. Therefore in the scalene case this is the only relation we need to consider. Even in the isosceles and equilateral cases the additional equivalences are small in number being determined by an action of a subgroup of Σ_3. See Remark 6.*

For anti-conformal equivalences we have a similar theorem.

Theorem 19 *Let notation be as in Theorem 12 above. Then there is an anti-conformal equivalence $h : S \to S'$ commuting with the group actions of G if and only if:*

1. *l, m, n are distinct and there is an $\omega \in \mathrm{Aut}(G)$ such that*
$$(a', b', c') = \omega \cdot (a^{-1}, b^{-1}, bc^{-1}b^{-1}),$$
2. *$l = m \neq n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a^{-1}, b^{-1}, bc^{-1}b^{-1}),$$
$$(a', b', c') = \omega \cdot (b^{-1}, a^{-1}, c^{-1})$$
holds,
3. *$l \neq m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a^{-1}, b^{-1}, bc^{-1}b^{-1}),$$
$$(a', b', c') = \omega \cdot (a^{-1}, c^{-1}, b^{-1})$$
holds,
4. *$l = n \neq m$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a^{-1}, b^{-1}, bc^{-1}b^{-1}),$$
$$(a', b', c') = \omega \cdot (c^{-1}, b^{-1}, a^{-1})$$
holds,
5. *$l = m = n$ and there is an $\omega \in \mathrm{Aut}(G)$ such that at least one of*
$$(a', b', c') = \omega \cdot (a^{-1}, b^{-1}, bc^{-1}b^{-1}),$$
$$(a', b', c') = \omega \cdot (b^{-1}, a^{-1}, c^{-1}),$$
$$(a', b', c') = \omega \cdot (a^{-1}, c^{-1}, b^{-1}),$$
$$(a', b', c') = \omega \cdot (c^{-1}, b^{-1}, a^{-1}),$$
$$(a', b', c') = \omega \cdot (b^{-1}, c^{-1}, ca^{-1}c^{-1}),$$
$$(a', b', c') = \omega \cdot (c^{-1}, a^{-1}, ab^{-1}a^{-1}).$$
holds.

2.3 More on the connection to Fuchsian groups

Let $\widetilde{\Delta_0}$ be a triangle in the hyperbolic plane with angles of measure $\frac{\pi}{l}$ radians, $\frac{\pi}{m}$ radians, and $\frac{\pi}{n}$ radians. Such a triangle exists and is unique up to isometry as long as $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. Let $\widetilde{p}$, $\widetilde{q}$, and $\widetilde{r}$ be the reflections in the sides of $\widetilde{\Delta_0}$ and let all other terminology be analogously defined to the surface case previously discussed. Let $\Lambda^* = T^*_{l,m,n} = \langle \widetilde{p}, \widetilde{q}, \widetilde{r} \rangle$, $\Lambda = T_{l,m,n} = \left\langle \widetilde{a}, \widetilde{b}, \widetilde{c} \right\rangle$. The groups $T^*_{l,m,n}$ and $T_{l,m,n}$ have presentations

$$T^*_{l,m,n} = \left\langle \widetilde{p}, \widetilde{q}, \widetilde{r} : \widetilde{p}^2 = \widetilde{q}^2 = \widetilde{r}^2 = (\widetilde{p}\widetilde{q})^l = (\widetilde{q}\widetilde{r})^m = (\widetilde{r}\widetilde{p})^n = 1 \right\rangle \text{ and}$$
$$T_{l,m,n} = \left\langle \widetilde{a}, \widetilde{b}, \widetilde{c} : \widetilde{a}^l = \widetilde{b}^m = \widetilde{c}^n = 1 \right\rangle.$$

Now there is a surface S with an (l,m,n)-rotation group G if and only if there is an exact sequence

$$\Gamma \hookrightarrow \Lambda \overset{\eta}{\twoheadrightarrow} G.$$

In this case Γ is a torsion free subgroup, $S = \mathbb{H}/\Gamma$, and the triple $(a,b,c) = (\eta(\widetilde{a}), \eta(\widetilde{b}), \eta(\widetilde{c}))$ is an (l,m,n)-generating vector for the G-action. Now Λ^* generates an (l,m,n)-tiling on $\mathbb{H}$, namely $\widetilde{\mathcal{T}} = \Lambda^*\widetilde{\Delta_0}$. This is the point where the Poincaré Polygon theorem is required to prove the tiling exists and that Λ^* acts simply transitively on $\widetilde{\mathcal{T}}$. As $\widetilde{\mathcal{T}}$ is Γ-invariant, then $\widetilde{\mathcal{T}}$ projects to a tiling $\mathcal{T}$ on S, whose master tile Δ_0 is the image of $\widetilde{\Delta_0}$. The tiling $\mathcal{T}$ will be kaleidoscopic if and only if $\Gamma \lhd \Lambda^*$. For then we may extend η

$$\Gamma \hookrightarrow \Lambda^* \overset{\eta}{\twoheadrightarrow} G^*.$$

and the images $p = \eta(\widetilde{p})$, $q = \eta(\widetilde{q})$, and $r = \eta(\widetilde{r})$) will be the global extensions of the local reflections in the sides of Δ_0. The automorphism θ is the automorphism induced by $\mathrm{Ad}_{\widetilde{q}} : \Lambda^* \to \Lambda^*$, $g \to \widetilde{q}\gamma\widetilde{q}$.

All the additional propositions and theorems can be proven by working with Λ, Λ^*, η, and the map $\mathbb{H} \longrightarrow \mathbb{H}/\Gamma$.

3 Organization of the classification process

Before outlining the approach to classification let us make a number of observations and collect some known results that will help shape our procedure.

Theorem 20 *Let G be a rotation group with a generating (l,m,n)-triple, acting on a surface of genus σ. Then:*

$$|G| \le 84(\sigma-1), \tag{11}$$

$$l,m,n \text{ divide } |G| \tag{12}$$

$$l,m,n \le 4\sigma+2, \tag{13}$$

and if p is a prime divisor of $|G|$ then

$$p = 2\sigma + 1 \text{ or } p \leq \sigma + 1. \tag{14}$$

The result 11, from [7], bounds the order of a rotation group, and statements 12 (which is trivial), and 13 and 14, (which may be found in [6]), restrict the orders of the automorphisms of a rotation group.

A rotation group for which $|G| = 84(\sigma - 1)$ is called a *Hurwitz group.* By calculations using the Riemann-Hurwitz equation, it is seen that if G is a Hurwitz group, then $(l, m, n) = (2, 3, 7)$. It is known that a Hurwitz group exists for $\sigma = 3$, 7, 14, and infinitely many higher genera but does not exist for $\sigma = 2, 4, 5$, 6 or 8-13 [8]. After Hurwitz groups, the largest value for

$$\frac{|G|}{\sigma - 1} = \frac{2}{1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)}$$

of 48 is achieved for $(l, m, n) = (2, 3, 8)$ and so the largest non-Hurwitz group order is given by $|G| = 48(13 - 1) = 576$. Thus if the Hurwitz groups were investigated separately the Magma/GAP database could be used to compute all remaining groups for several more genera. The present paper picked 13 as a convenient stopping point, since in this case the group order is automatically divisible by $12 = \sigma - 1$ leading to many interesting solvable groups. On the other hand genus 17 and $(2, 3, 8)$ would lead to the groups of order 768 which have not been enumerated.

3.1 Classification outline

Now suppose the genus σ is fixed. Here are the steps of the algorithm.
Step 1. Construct a list of feasible $|G|$ and (l, m, n).
Step 2. For each entry in the list found in Step 1, determine the possible structures of the group G, (e.g., is the group cyclic) or limitations on the group structure which reduce the number of groups in the Magma/GAP database to be investigated. Examples of such shortcuts are give in section 4.
Step 3. Given a G and (l, m, n) find representatives of the $\mathrm{Aut}(G)$-classes of (l, m, n)-triples, (if any) using an "approximation" to the automorphism group as an intermediate step.
Step 4. Determine which triples are equivalent under the transformations given in Theorems 17 and 19.
Step 5. For each representative class found in Step 4 determine if the automorphism θ exists.

To begin classification for a specific genus, we find all possible values for $|G|$ and (l, m, n) by using the Riemann-Hurwitz equation 6. The corresponding vectors $(|G|, l, m, n)$, are called *branching data* and have been enumerated in a Maple script `bradatpoly.mws` available at the website [17]. So we start with this list of data, and we must eliminate those for which no group exists, and find all groups that satisfy the feasible data. The conditions 12, 13 and 14, which limit the possible l, m, n, and $|G|$, are easily implemented in the Maple computer search for branching data so we remove these infeasible $(|G|, l, m, n)$ from the list of branching data right at the beginning.

3.2 Representing groups and automorphism groups

Step 2 is the most difficult and requires some mathematical cleverness unless one is to rely on the database. Even though computer calculation makes our computations very quick, the number of possible groups of a given order grows very quickly. In particular if we need to consider the dreaded 2-group case we might have to consider 56092 groups in the case $|G| = 256$. Thus we need to find some good workable methods to limit the structure of the groups considered. We consider this in the next section. In the remainder of this section we consider how to organize the calculations so that they may be rapidly calculated using the Magma/GAP data base.

Since we will be using Magma we need to consider how we will represent the group. Magma and GAP have several categories of finite groups and several ways of representing them. The ones that will be interesting to us are:

- cyclic groups
- abelian groups,
- power conjugate representations of solvable groups, and
- permutation groups,

The above classifications are each interesting in their own right and the computer calculation is encoded separately for each class, except that the non-abelian solvable cases is split into a p-group case and a non p-group case. Each of these representations has various strengths and limitations.

- Most of the abelian cases are cyclic, and there is already extensive information on these groups.
- Permutation groups can be universally used and are very fast in computation.
- Both permutation and cyclic groups allow calculation of a "quick and dirty" approximate automorphism group, described below.
- Power-conjugate representations of solvable groups will apply to most of our solvable groups, since that is the way that the data base is constructed.

Another difficulty we face is the fact that the automorphism group is not always easily or efficiently calculated. However, we will not necessarily need the full automorphism group, we will need only to directly construct certain automorphisms. However, we will find it useful to have an "approximate automorphism group". The notion follows from the following proposition whose proof is straight forward.

Proposition 21 *Let $G \subseteq \Sigma$ be a pair of groups. Let $N = N_\Sigma(G)$ and $Z = Z_\Sigma(G)$ be the normalizers and centralizers of G in Σ respectively. Let $Ad : N \to \mathrm{Aut}(G)$ be the adjoint homomorphism defined by $Ad_a(g) = aga^{-1}$. Then Ad maps N/Z isomorphically onto a subgroup M of $\mathrm{Aut}(G)$ containing the inner automorphisms of G,*

$$\mathrm{Inn}(G) \subseteq M \subseteq \mathrm{Aut}(G)$$

We will call M an *approximate automorphism group* of G if M is of low index in $\mathrm{Aut}(G)$. If we embed G in a permutation group by its Cayley representation we can recover the automorphism group exactly (see [5] or [9]).

Proposition 22 *Let Σ_G be the group of permutations of the elements of G. For $g \in G$ let $L_g, R_g \in \Sigma_G$ be the left and right regular representations defined by $L_g(x) = gx$, $R_g(x) = xg^{-1}$. Let $G_L = \{L_g : g \in G\}$, let $N = N_{\Sigma_G}(G_L)$ and $A = \{h \in N : h(1) = 1\}$ be the stabilizer of the identity. Then*

- $G_L \backsimeq G,\ A \backsimeq \mathrm{Aut}(G)$
- $N = A \ltimes G_L$ *an internal semi-direct product, and hence*
- $N \backsimeq \mathrm{Aut}(G) \ltimes G = \mathrm{Hol}(G)$, *the holomorph of G.*

Similar results hold for the right regular representation.

Remark 23 *Proposition 21 is only useful if we have a faithful permutation representation of low degree, in which case we may take Σ to be the symmetric group, or if normalizers and centralizers are easily calculated. For example, the cyclic group $\mathbb{Z}_n$ may be embedded in Σ_n by using an n-cycle, in this case we recover $\mathrm{Aut}(G)$ exactly because of proposition 22. If the representation is of low degree there is a chance that N can be computed efficiently, though in practice it was found that if the permutation representation had multiple orbits, it was better to take $\Sigma = G$. For non-abelian solvable groups $M = \mathrm{Inn}(G)$, i.e., $\Sigma = G$, is often a good approximation already.*

The approximate automorphism groups can give us a finer approximate classification of triples, from which we need to select representatives. Later, in section 3.4 we show how to construct automorphisms with specific action on generators, such as the automorphism θ. This techniques can be used to refine the approximate classification into an exact classification.

3.3 Finding generating triples

Assume that we know G and (l, m, n). Also assume that we have G embedded in some overgroup Σ and adopt the notation of Proposition 21 above. We define several objects: Let

$$O_G(s) = \{g \in G : o(g) = s\}.$$

This object is a union of several conjugacy classes of G. Also define

$$X_G(l,m,n) = \{(a,b,c) \in G^3 : o(a) = l, o(b) = m, o(c) = n, abc = 1\}$$
$$= \{(a,b,c) \in O_G(l) \times O_G(m) \times O_G(n) : abc = 1\}$$

and

$$X_G^o(l,m,n) = \{(a,b,c) \in X_G(l,m,n) : G = \langle a,b,c\rangle\}.$$

Remark 24 *The cardinality of the first set can sometimes be easily calculated means of character theory, see examples in [2]. If $X_G(l,m,n) = X_G^o(l,m,n)$ then the number of equivalence classes is easily calculated.*

The set $X_G^o(l,m,n)$, or rather representatives of the $\mathrm{Aut}(G)$-equivalences classes of this set is what we are seeking. Observe that $\mathrm{Aut}(G)$ acts freely on $X_G^o(l,m,n)$ since an automorphism fixing the elements of a generating set must be the identity. Thus as a "first approximation" we have

$$\#\text{ rotation groups } = \frac{|X_G^o(l,m,n)|}{|\mathrm{Aut}(G)|}.$$

Since $\mathrm{Aut}(G)$ is fairly large if G has any complexity, direct calculation of $X_G^o(l,m,n)$ as a subset of $O_G(l) \times O_G(m) \times O_G(n)$ is expensive. We will therefore work with representatives. We may not have $\mathrm{Aut}(G)$ available but at least we have the action of $M = N/Z$, via the action of N. Each $\mathrm{Aut}(G)$-orbit on $X_G(l,m,n)$ consists of several M-orbits. We may construct representatives $\mathrm{Aut}(G)$-orbit on $X_G^o(l,m,n)$ by refining Step 3 above into three substeps:
Step 3.1 Construct representatives of the N-orbits on $X_G(l,m,n)$.
Step 3.2 Determine which of the constructed representatives generate G.
Step 3.3 Find out which representatives constructed above are $\mathrm{Aut}(G)$-equivalent.

Consider two triples $(a,b,c), (a',b',c')$ in $X_G(l,m,n)$. Since $c = (ab)^{-1}$ then

$$X_G(l,m,n) = \{(a,b,c) : (a,b) \in O_G(l) \times O_G(m),\ o(ab) = n\},$$

and $(a',b',c') = \omega\cdot(a,b,c)$ if and only if $(a',b') = \omega\cdot(a,b)$. Thus we can construct representatives of the N-orbits on $X_G(l,m,n)$ by constructing representatives of N-orbits on $O_G(l) \times O_G(m)$ and then determining which of these determine representatives of $X_G(l,m,n)$. Of course we may use (a,c) or (b,c) if this results in a smaller number of calculations. The following proposition describes the action of N on $O_G(l) \times O_G(m)$.

Proposition 25 *Let G, Σ, N, and $O_G(l) \times O_G(m)$ be as above. Let $a_1, \ldots, a_s$ be a set of representatives of the N-action on $O_G(l)$. For each a_i let $Z_i = Z_N(a_i)$ be the centralizer of a_i. Let $\{b_{i,1}, \ldots, b_{i,s_i}.\}$ be a complete set of representatives of Z_i-action on $O_G(m)$. Then, the set $\{(a_i, b_{i,j_i}) : 1 \le i \le s, 1 \le j_i \le s_i\}$ is a complete set of representatives for the N-action on $O_G(l) \times O_G(m)$.*

Proof. The projection $\pi : O_G(l) \times O_G(m) \to O_G(l)$ given by $(a,b) \to a$ is an N-equivariant surjection. Therefore, the pre-image of each N-orbit in $O_G(l)$ is a union of N-orbits in $O_G(l) \times O_G(m)$. Now let a_i be one of the representatives and let $\mathcal{O}_1, \ldots, \mathcal{O}_t$ be the N-orbits lying above the orbit Na_i, i.e., $Na_i \times O_G(m) = \bigcup_{j=1}^{t} \mathcal{O}_i$. Now (a_i, b_{i,j_i}) be an element of $\pi^{-1}(a_i) \cap \mathcal{O}_j = a_i \times O_G(m) \cap \mathcal{O}_j$. Since $\mathcal{O}_j$ is an orbit then any element (a_i, b) in it satisfies $(a_i, b) = \omega \cdot (a_i, b_{i,j_i})$ for some $\omega \in N$. But $\omega \cdot a_i = a_i$ so $\mathcal{O}_j = a_i \times Z_i b_{i,j_i}$. The Proposition now follows from the fact that $O_G(m) = \bigcup_{j=1}^{t} Z_i b_{i,j_i}$, a disjoint union of $t = s_i$ orbits.

Thus we may now construct a complete set of representatives of the N-action on $X_G(l, m, n)$ as the following set.

$$\left\{ (a_i, b_{i,j_i}, (a_i b_{i,j_i})^{-1}) : 1 \leq i \leq s, 1 \leq j_i \leq s_i, o\,(a_i b_{i,j_i}) = n \right\}.$$

This solves Step 3.1 above. Step 3.2 is pretty straight forward since Magma can easily calculate the size of a group generated by elements. Step 3.3 can be implemented through our discussion on automorphism construction which we take up next.

3.4 Constructing specific automorphisms of G

Our problem of constructing θ can be generalized in a way that is helpful to us. Consider the following situation. Let $G = \langle x_1, \ldots, x_t \rangle$ and let $y_1, \ldots, y_t$ be a set of elements of G. Can we construct an automorphism $\theta \in \mathrm{Aut}(G)$ such that $\theta(x_i) = y_i$? Clearly our extension problem is simply the case $x_1 = a$, $x_2 = b$ and $y_1 = a^{-1}$, $y_2 = b^{-1}$. Suppose that we have defined a bijective map $\theta : G \to G$ such that $\theta(x_i) = y_i$ for all i. Then, when does $\theta \in \mathrm{Aut}(G)$? This is easily disposed of in the next proposition.

Proposition 26 *Let $G = \langle x_1, \ldots, x_t \rangle$ be a finite group, and Σ_G and L_g for $g \in G$ be defined as in Proposition 22. Let $\theta \in \Sigma_G$ be a bijective mapping of G satisfying $\theta(x_i) = y_i$ of all i, where y_i is a sequence of elements of G. Then $\theta \in \mathrm{Aut}(G)$ if and only if*

$$\theta L_{x_i} \theta^{-1} = L_{y_i} \text{ for all } i, \tag{15}$$

and

$$\theta(1) = 1.$$

Proof. If $\theta \in \mathrm{Aut}(G)$, then for each $g \in G$

$$\theta L_{x_i} \theta^{-1}(g) = \theta(x_i \theta^{-1}(g)) = \theta(x_i) g = y_i g.$$

Thus $\theta L_{x_i} \theta^{-1} = L_{y_i}$. This proves necessity of the conclusion. Now lets prove sufficiency. The hypothesis shows that $\theta \in \Sigma_G$ normalizes the subgroup $G_L = \{L_g : g \in G\}$. Thus $Ad_\theta : L_g \to \theta L_g \theta^{-1}$ is an automorphism of G_L onto its image, which is also G_L. It follows that there is a bijective map $\psi : G \to G$ satisfying

$$\theta L_g \theta^{-1} = L_{\psi(g)}, \text{ or alternatively}$$
$$\theta(gx) = \psi(g)\theta(x) \text{ for all } g \text{ and } x.$$

However, $L : G \to G_L$ is an isomorphism, and by definition $\psi = L^{-1} \circ Ad_\theta \circ L$. Thus we have proven that ψ is a automorphism of G. We finish by noting that $\theta(g) = \theta(g \cdot 1) = \psi(g)\theta(1) = \psi(g)$, by hypothesis and previous calculation. $\blacksquare$

Remark 27 *Note that the assumption $\theta(1) = 1$ is necessary since θL_z also satisfies the hypotheses 15 when $z \in Z(G)$.*

We have now solved the problem when θ is globally defined, but we only start off with $\theta(x_i) = y_i$. How do we construct θ? Suppose for the sake of argument that θ does extend to an automorphism and satisfies the homomorphism property. We can try to build the graph of θ inductively, based on the idea that $\theta(x_i x_j) = \theta(x_i)\theta(x_j) = y_i y_j$. Since $G = \langle x_1, \ldots, x_t \rangle$ we should be able to completely build the graph of θ inductively as follows. Set $W_0 = \{(1,1)\} = \{(1, \theta(1))\}$ and set $x_0 = 1$, $y_0 = 1$. Having defined W_n, define W_{n+1} by

$$W_{n+1} = \bigcup_{i=0}^{t} \{(x_i x, y_i y) : (x, y) \in W_n\}.$$

Since $x_0 = 1$ and $y_0 = 1$ then $W_n \subseteq W_{n+1}$. Assume as an inductive hypothesis that W_n has the form $\{(x, \theta(x)) : (x, y) \in W_n\}$, i.e. it is a partial graph of θ. Now each element of W_{n+1} has the form

$$(x_i x, y_i y) = (x_i x, \theta(x_i)\theta(x)) = (x_i x, \theta(x_i x))$$

and so W_{n+1} is a bigger portion of the graph of θ. We terminate the process when $|W_n| \geq |G|$. Now given just the sequences $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ the sets W_n can all be constructed. Note that eventually that $|W_n| \geq |G|$ since every $x \in G$ eventually occurs in a pair (x, y) in some set. Let W_N be the first such set. Now, a couple of bad things can happen if θ doesn't extend to an automorphism. We may have (x, y_1), $(x, y_2) \in W_N$ with $y_1 \neq y_2$. However this can only happen if $|W_N| > |G|$ or $|W_N| = |G|$ and

$$|\{x : (x, y) \in W_N, \text{ for some } y\}| < |G|. \tag{16}$$

If the first bad thing doesn't happen then the other bad thing that can happen is that the map defined by W_N is not $1-1$. This will only happen if

$$|\{y : (x, y) \in W_N, \text{ for some } x\}| < |G|. \tag{17}$$

Thus with a few simple tests we can determine if θ extends to some bijective map of G. If we get this far then we can apply tests in Proposition 26. Let us formalize the preceding discussion in a Proposition.

Proposition 28 *Let $G = \langle x_1, \ldots, x_t \rangle$ be a finite group and let $y_1, \ldots, y_t$ be sequence of elements of G. Let $W_0, \ldots, W_N$ be the sets constructed above. Suppose that $|W_N| = |G|$ and that neither of the conditions 16 or 17 occur. Then W_N is the graph of a bijective map $\theta : G \to G$. Now further suppose that the conditions 15 hold. Then θ is the unique automorphism satisfying $\theta(x_i) = y_i$ for all i.*

Remark 29 *The existence of θ, the automorphisms called for in Step 3.3, and the automorphisms called for in establishing conformal or anti-conformal equivalence in Theorems 17 and 19 can be efficiently determined by our algorithm and there is no need to calculate the full automorphism group.*

4 Exploiting group structure

The Magma/Gap database has "only" 174,365 groups, even though it covers only groups of order less than 1000 with orders different from 512 and 768. Thus any way in which a potential group action can be eliminated will be useful. Moreover in constructing examples and counter-examples beyond the range of the data base, general group theoretic constructions will be important.

4.1 General considerations

Cyclic and abelian groups The easiest $(|G|, l, m, n)$ data to start with are those that have one of l, m, or n equal to $|G|$, for then the group must be cyclic. In [6], Harvey gives necessary and sufficient conditions on the branching data $(|G|, l, m, n)$ so that a cyclic rotation group exists. In particular, if G is cyclic, then lcm of any two of l, m, and n must equal G. In each of the cyclic cases the group calculation is done as suggested in Remark 23.

To check for possible abelian groups, it is useful to remember that all of the groups must be generated by two elements. Thus the group must be of the form $\mathbb{Z}_p \times \mathbb{Z}_q$ where $pq = |G|$ and $q|p$ because of the fundamental theorem on abelian groups. Furthermore, because the order of the product of two elements in an abelian group divides the lcm of the orders of the constituent factors any one of l, m, and n must divide the lcm of the other two.

Sylow contradictions Occasionally the Sylow theorems may be applied to show that certain actions are impossible. In the computer search this information was not directly used, rather there was no corresponding group in the data base because of the Sylow contradiction.

Example 30 *Consider the branching data $\sigma = 5,\ 30, (3,5,5)$ for a possible rotation group G. By Sylow's theorems, a group of order 30 must have 1 or 10 3-Sylow subgroups, and 1 or 6 5-Sylow subgroups. If we assume that neither of these numbers is 1, then we see that our group has more than 30 elements. Therefore, one of these Sylow subgroups must be normal. However, this would mean the first two elements in the generating triple would generate 15 elements instead of all 30. Therefore, there is no rotation group corresponding to this data.*

Solvable group structure Every solvable group has a power conjugate or polycyclic (PC) representation. This means we have

$$G = \langle a_1, a_2, a_3, ..., a_n \rangle .$$

for some generators, $a_1, a_2, ..., a_n$ with special properties. Let $G_i = \langle a_i, \ldots, a_n \rangle$, then automatically $G = G_1 \supset G_2 \supset \cdots \supset G_{n+1} = \{e\}$, and we may, WLOG, assume a proper inclusion at each stage. Because G is solvable, the generators may be chosen so that we may assume something stronger $G = G_1 \rhd G_2 \rhd \cdots \rhd G_{n+1} = \{e\}$. As a consequence for $a_j^{a_i} \in G_{i+1}$ for all $j > i$. In addition, if we assume we assume that the (proper) chain above is long as possible, then $|G_i/G_{i+1}| = p_i$, a prime, for $i = 1, \ldots, n$. It then follows that every element of G may be written in a unique form

$$g = a_1^{r_1} a_2^{r_2} \cdots a_n^{r_n}, \tag{18}$$

where $r_1 \leq p_1, \ldots, r_n \leq p_n$. In particular, the elements of G_i have the form $a_i^{r_i} a_{i+1}^{r_{i+1}} \cdots a_n^{r_n}$. Thus the relations for G all have the form

$$\begin{aligned} a_i^{p_i} &= w_i = w_i(a_{i+1}, \ldots, a_n) \in G_{i+1} \\ a_j^{a_i} &= w_{i,j} = w_{i,j}(a_{i+1}, \ldots, a_n) \in G_{i+1}, \ j > i. \end{aligned}$$

The words w_i and $w_{i,j}$ determine a PC presentation

$$G = \left\langle a_1, a_2, ..., a_n \mid a_i^{p_i} = w_i, a_j^{a_i} = w_{i,j} \right\rangle.$$

The number n is determined by the factorization of $|G|$ into primes. For example, here are the non-trivial examples up to order 6

$$\begin{aligned} \mathbb{Z}_4 &= \left\langle a_1, a_2 \mid a_1^2 = a_2, a_2^2 = 1, a_2^{a_1} = a_2 \right\rangle, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 &= \left\langle a_1, a_2 \mid a_1^2 = 1, a_2^2 = 1, a_2^{a_1} = a_2 \right\rangle, \\ \mathbb{Z}_6 &= \left\langle a_1, a_2 \mid a_1^2 = 1, a_2^3 = 1, a_2^{a_1} = a_2 \right\rangle, \\ \Sigma_3 &= \left\langle a_1, a_2 \mid a_1^2 = 1, a_2^3 = 1, a_2^{a_1} = a_2^2 \right\rangle, \end{aligned}$$

and two examples of order 12.

$$\begin{aligned} A_4 &= \left\langle a_1, a_2, a_3 \mid a_1^3 = 1, a_2^2 = 1, a_3^2 = 1, a_2^{a_1} = a_3, a_3^{a_1} = a_2 a_3, a_3^{a_2} = a_3 \right\rangle, \\ \mathbb{Z}_{12} &= \left\langle a_1, a_2, a_3 \mid a_1^3 = a_2, a_2^2 = a_3, a_3^2 = 1, a_2^{a_1} = a_2, a_3^{a_1} = a_3, a_3^{a_2} = a_3 \right\rangle, \end{aligned}$$

The virtue of exploiting the PC structure of a group is that it allows rapid calculation and gives a universal representation and computation format for all solvable groups. Since in the small group database, discussed below, over 99.88% of the groups are solvable, this representation is of great value.

There are other structural properties and invariants that may be computed quickly from the PC presentation. Two of these are the exponent and the abelianization of the group. The group G has elements of order l, m, and n, therefore the exponent of G must be divisible by each of l, m and n. If the exponent of G fails the divisibility test then it may be rejected before actually committing to calculating steps 3, 4, and 5 of the classification process.

Let us discuss the abelianization in a bit more detail, as it has greater consequences for classification and is strongly linked to the PC presentation. Our group $G = \langle a, b \rangle$ is generated by two elements, and so every homomorphic image of G, in particular its abelianization, must be generated by two or fewer elements. The minimal number of generators in the abelianization can be quickly computed from the PC format. Indeed, let $g \to \overline{g}$ be the abelianization map $G \to G_{ab}$ and write the typical abelianized element $\overline{g}$, with g as in equation 18, in additive format:

$$\overline{g} = r_1\overline{a}_1 + r_2\overline{a}_2 + \cdots + r_n\overline{a}_n.$$

The relations then become

$$\begin{aligned} p_i\overline{a_i} &= \overline{w_i} \in \mathbb{Z}\overline{a}_{i+1} + \cdots + \mathbb{Z}\overline{a}_n, \text{ or} \\ p_i\overline{a_i} - \overline{w}_i &= 0 \\ \overline{a_j} &= \overline{w_{i,j}} = \mathbb{Z}\overline{a}_{i+1} + \cdots + \mathbb{Z}\overline{a}_n, \text{ or} \\ \overline{a_j} - \overline{w_{i,j}} &= 0. \end{aligned}$$

We may write the relations in the form $M\overline{A} = 0$ where $\overline{A} = \begin{bmatrix} \overline{a}_1 & \overline{a}_2 & \cdots & \overline{a}_n \end{bmatrix}^t$ and M is a matrix of integers. Each relation gives rise to a row of the matrix yielding $n(n+1)/2$ rows. Rows of zeros correspond to abelian commutation relations $a_j^{a_i} = a_j$ or $\overline{a_j} = \overline{a_j}$. Subject to proper ordering, the matrix determines G. For example the matrix equations for A_4 and $\mathbb{Z}_{12}$ are

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_2 \\ \overline{a}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_2 \\ \overline{a}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

respectively. Now we may put M into Smith normal form, i.e., there are square, invertible integral matrices U, V such that UMV is a diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ such that $d_i | d_{i+1}$. The Smith normal forms of the two matrices above are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easily shown that the minimal number of generators of G_{ab}, is equal to the number of d_i different from 1. The method may be easily adapted to any PC presentation and the form of the PC presentation makes the reduction to normal from fairly easy. Now the abelianization test would not eliminate either of these groups – indeed they both support triangular actions. However many are eliminated as we see in the next subsection.

Remark 31 *If we are looking at tilings by (s,t,u,v) quadrilaterals then we seek groups $G = \langle a,b,c,d \rangle$ such that $o(a) = s$, $o(b) = t$, $o(c) = u$, $o(d) = v$, $abcd = 1$, $|G| = (2\sigma - 2)/(2 - 1/s - 1/t - 1/u - 1/v)$ and there is a θ satisfying $\theta(a) = a^{-1}$, $\theta(b) = b^{-1}$. In this case the minimal number of generators of G is three or less. analogous statements can be made for polygons of larger size. Thus the two generator requirement is artificial unless we restrict our attention to triangles.*

The PC format also works well with respect to an inductive classification of rotation groups. Indeed as noted in the next section, most rotation groups are solvable. Now suppose we wish to find solvable rotation groups of a given order and given branching data (l,m,n). The group G has a normal subgroup N such that G/N is solvable and acts on the surfaces S/N of lower genus. Furthermore G/N is a rotation group on S/N generated by the images $\overline{a}$, $\overline{b}$ and $\overline{c}$ in G/N, and branching data $(\overline{l},\overline{m},\overline{n})$, where $\overline{l},\overline{m}$, and $\overline{n}$ divide l, m, and n, respectively. ($(\overline{l},\overline{m},\overline{n})$ need not define a hyperbolic triangle). In any event the G/N may be assumed to have been previously classified because of the lower order and genus. Now fix a G/N and a $(\overline{l},\overline{m},\overline{n})$. The group G/N will have a *PC* generating set of the form $\overline{a_1}, \overline{a_2}, ..., \overline{a_n}$, and let $a_{n+1}, a_{n+2}, \ldots, a_m$ denote a PC generating set for N. Since N is normal, any set of lifts $a_1, a_2, \ldots, a_n$ of $\overline{a_1}, \overline{a_2}, ..., \overline{a_n}$ to G may be prepended to $a_{n+1}, a_{n+2}, \ldots, a_m$ to form a *PC* generating set. Moreover the relations coming from G/N are obtained by removing the overbars and adding words in $a_{n+1}, a_{n+2}, \ldots, a_m$ to the right hand side of the $\overline{w_i}$ and $\overline{w_{i,j}}$. In the matrix formulation we insert new columns on the right of the matrix and then additional rows from the presentation of N. Now the fact that G is a rotation group may restrict the way in which we can add new variables and relations to the presentation of G/N. An example of geometric significance, with the simplest extension problem is the following. Suppose that S is a surface of genus σ with an involution ι. Let now $N = \langle \iota \rangle$, and ask what are the possible solvable groups that extends the action of N. In the well known case of hyperelliptic involutions, S/N is a sphere and G/N is one of the rotation groups of the sphere.

4.2 The small group database

The small group data base contains the following 174,365 groups: all groups of order $n \leq 1000$, except those of order 512 and 768. All solvable groups are given in power conjugate format. The remaining 208 groups are given as permutation groups. This forces G to be divisible by 60, 168, 360, or 504 since we are only considering groups of order ≤ 1000, and there must be some simple quotient in a composition series for G. This severely limits G, and the non-solvable case is actually computed rather quickly. There are conditions which force G to have a simple quotient. If l, m and n are distinct primes, e.g., $2, 3$ and 7 then an elementary argument shows that G cannot have an abelian quotient. Thus it must have a simple non-abelian quotient. This is true of the Hurwitz surfaces with $(2,3,7)$ tilings, that added to their maximal symmetry, have made them very interesting.

As hinted previously, it is actually feasible to easily calculate the abelianization of the solvable groups (using the Magma abelianization command) we get the following results.

# generators of G_{ab}	1	2	3	4	5	6
# groups in data base	4790	17796	47010	69163	34788	592
% of groups	2.75	10.21	26.96	36.97	19.95	0.34

# generators of G_{ab}	7	8	Permutation Groups	Total
# groups in data base	17	1	208	174,365
% of groups	0.01	0.00	0.12	100

Thus it appears that about 87% of the groups are automatically ineligible, and the search is cut down to a much smaller number of groups. Even so, only a tiny fractions of the groups tested yield any generating triples. The total search is not long and takes only a few hours. The paucity of results suggest that any further enumerations of rotation groups should perhaps follow an inductive construction described above rather than the bludgeoning the classification into shape with Magma. Indeed this would seem to be the most appropriate way to undertake classifying rotation groups of order 512, since the 2 generator condition may provide the great reduction required to make this case manageable.

5 Examples and counter-examples

Metacyclic groups Metacyclic groups, especially split metacyclic groups, are a good source of examples and counter-examples, especially when constructing infinite families. A general class of such examples is discussed in [2]. Rather than reviewing the entire theory we give a simple example. Generalizing the dihedral group D_n we define the split metacyclic group of order pq

$$D_{p,q,r} = \langle x, y : x^p = y^q, y^x = y^r \rangle, \text{ where } r^p \equiv 1 \bmod q.$$

Example 32 *Consider the data* $\sigma = 4,\ |G| = 32, (l, m, n) = (2, 4, 16)$*. Suppose* G *is a group corresponding to this data. Then we know that* G *must have a cyclic subgroup of order* 16 *since* $n = 16$*. This subgroup must be normal in* G *since its index is* 2*. Since* $l = 2$*, and any two elements in the generating triple must generate all of* G*, then there must be an element of order* 2 *not contained in the cyclic subgroup of order* 16 *(otherwise only* 16 *elements are generated by the first and third elements in the generating triple). Therefore we must have* $G \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{16}$*. We may construct all such groups by finding all elements in* $Aut(\mathbb{Z}_{16}) \cong \mathbb{Z}_{16}^*$ *of order* 2*. Thus, setting* $x = a^{-1}$ *and* $y = c^{-1}$*,* G *must be one of* $D_{2,16,r} = \langle x, y : x^2 = y^{16}, y^x = y^r \rangle$*, where* $r = 1, 7, 9,$ *or* 15*. However,* $b = a^{-1}c^{-1} = xy$ *must have order* 4 *and hence* $xyxy = y^x y = y^{r+1}$ *has order* 2*. The only possibility is* $r = 7$*. Observe that* $\theta : x \to x$*, and* $y \to y^9$ *is an* *automorphism satisfying*

$$\theta(a) = \theta(x) = x = a^{-1},$$
$$\theta(b) = \theta(xy) = xy^9 = xy^9xx^{-1} = y^{-1}x^{-1} = (xy)^{-1} = b^{-1}.$$

Thus the tiling on S is kaleidoscopic.

Odd order tiling groups In the initial part of the investigation of surfaces of genus 2,3,4, and 5, [12] two observations emerged. First, there is only one non-abelian odd order rotation group, the non-abelian group of order 21, and secondly there are no non-abelian odd order tiling groups in the given range. The following result shows that there are in fact infinitely many odd order non-abelian tiling groups.

Proposition 33 *Let $p > 2$ be a prime, and let G be the non-abelian group of order p^3 such that all elements of G have order p. Then G is a tiling group of a surface of genus $\sigma = \frac{1}{2}(p^3 - 3p^2 + 2) = 1 + \frac{1}{2}p^2(p-3)$.*

Proof. The group G has the presentation $\langle x, y \mid x^p = y^p = 1, [x,y]^x = [x,y]^y = [x,y]\rangle$. It is also known that the group G is *extra-special*, that is, the center of G, $Z(G)$, and the derived group of G, $G' = [G,G]$, coincide (see [9]). Since all elements of G have order p, then we must have $(l,m,n) = (p,p,p)$. Consider the generating triple $(x, y, (xy)^{-1})$. We need an automorphism θ of G such that $\theta(x) = x^{-1}$, $\theta(y) = y^{-1}$, and $\theta^2 = id$. In order for this to be an automorphism, the relations in the presentation still must hold after applying θ. First note that

$$[x,y]^x = x^{-1}[x,y]x = [x,y] \iff$$
$$[x,y]^{x^{-1}} = x[x,y]x^{-1} = [x,y].$$

Therefore in G we have $[x,y]^{x^{-1}} = [x,y]^{y^{-1}} = [x,y]$. Now in order for θ to be an automorphism, we must have $[x,y] = [x^{-1}, y^{-1}]$. We have,

$$[x,y] = [x^{-1}, y^{-1}] \iff x^{-1}y^{-1}xy = xyx^{-1}y^{-1} \iff x^{-1}y^{-1}xyyx = xy.$$

Notice that $x^{-1}y^{-1}xy \in G' = Z(G)$, so that this element commutes with xy. So now $(x^{-1}y^{-1}xy)(yx) = (yx)(x^{-1}y^{-1}xy) = xy$, and $[x^{-1}, y^{-1}]^{x^{-1}} = [x^{-1}, y^{-1}]^{y^{-1}} = [x^{-1}, y^{-1}]$. It is trivial that $x^{-p} = y^{-p} = 1$, and that $\theta^2 = id$, so θ is the desired automorphism. It follows that G is a tiling group, and the genus of the surface it tiles can be calculated using the Riemann-Hurwitz equation. ■

Remark 34 *For $p = 3$ the surface is a torus with $\sigma = 1$. The next smallest example is $p = 5$ and $\sigma = 26$, so the examples would not have shown up in our study.*

Non-tiling rotation groups. There is only one non-tiling rotation group in the tables for genus 4 and 5. This group of order 20 and the aforementioned group of order 21 are the only ones for $2 \leq \sigma \leq 5$, as discovered during the initial stage of the determination of groups. The following theorem characterizes an entire class of non-tiling symmetry groups.

Proposition 35 *Let G be a non-abelian group such that $|G| = pq$, where p and q are distinct odd primes. Then G is a rotation group, but is not a tiling group.*

Proof. By using Sylow's Theorem, we see that $p < q$, $G \cong \mathbb{Z}_p \ltimes \mathbb{Z}_q$, where $q \equiv 1(\text{mod } p)$, (possibly reversing the roles of p and q). By finding an element of $\text{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_q^*$, we obtain a presentation for G:

$$G \cong \langle x, y \mid x^p = y^q = 1, x^{-1}yx = y^{k_0} \rangle$$

where $k_0^p \equiv 1(\text{mod } q)$ and $k_0 \not\equiv 1(\text{mod } q)$. In fact, it can be easily shown that for any two elements $z, w \in G$ of orders p and q respectively that $z^{-1}wz = w^k$ where k is one of the $p-1$ non-trivial solutions to $k^p \equiv 1(\text{mod } q)$. Now, if G is to be (l, m, n)-generated, then we must use only p or q for l, m, and n, since if one is pq, then G is cyclic. We also know that G should be generated by two of the elements in our generating triple (a, b, c). But G has a unique normal Sylow subgroup of order q by Sylow's theorem, so if two of l, m, or n are q, then not all of G is generated by the two elements of order q. Therefore we must have $(l, m, n) = (p, p, q)$ or (p, p, p), or a permutation of these.

First let us deal with the case (p, p, q), and let (a, b, c) be a (p, p, q) generating triple. As noted above $a^{-1}ca = c^k$ for k as above. Now we have G is a symmetry group of a surface of genus $\sigma = \frac{1}{2}(q-1)(p-2)$ (by the Riemann-Hurwitz equation). Now assume that G is a tiling group, and that we have an automorphism ψ of G such that $\psi(a) = a^{-1}$, $\psi(b) = b^{-1}$, and $\psi^2 = id$. Remembering that $abc = 1$, we have $1 = \psi(1) = \psi(abc) = \psi(a)\psi(b)\psi(c) = a^{-1}b^{-1}\psi(c)$. Thus

$$\psi(c) = ba = a^{-1}c^{-1}a = c^{-k}. \tag{19}$$

Now we must have $\psi^2(c) = c$, while

$$\psi(\psi(c)) = \psi(c^{-k}) = (\psi(c))^{-k} = (c^{-k})^{-k} = c^{k^2}.$$

But now if $c^{k^2} = c$, then since $c^q = 1$, we have $k^2 \equiv 1(\text{mod } q)$. However, k is a non-trivial solution of $k^p \equiv 1(\text{mod } q)$. Since 2 and p are relatively prime then $k \equiv 1(\text{mod } q)$, a contradiction, and so such an automorphism ψ cannot exist. Therefore, G is not a (p, p, q) tiling group.

Now consider the case of $(l, m, n) = (p, p, p)$. Many examples of (p, p, p)-generating triples (a, b, c) exist, say $(x, xy, y^{-1}x^{p-2})$, for this symmetry group. The genus is $\frac{1}{2}(pq - 3q - 2)$. By considering the map $\mathbb{Z}_q \to G \to \mathbb{Z}_p$ and the images $\overline{a}, \overline{b}, \overline{c}$ of a, b, c in $\mathbb{Z}_p$ we see $\overline{b} = \overline{a}^s$ for some s. It follows that $b = da^s$ for some $d \in \mathbb{Z}_q$. By reusing a previous argument we get an equation similar to 19

$$\psi(d) = \psi(ba^{-s}) = b^{-1}a^s = b^{-1}d^{-1}b = d^{-k},$$

for a k as above, and we get another contradiction, finishing our proof. ■

The reader should note that the above theorem and proof may be adjusted to apply to some non-abelian groups of the form $\mathbb{Z}_4 \ltimes \mathbb{Z}_p$ with the specific triple $(4, 4, p)$, where $p > 3$ is prime, including the group of order 20 in the Table 6.3.a. This result is not as interesting since it is for a specific triple for a class of groups, and not for all possible triples. Here is a general statement for that covers all the examples above.

Proposition 36 *Let (a, b, c) be an (l, m, n)-generating vector for the rotation group G. Suppose that G has a characteristic, cyclic subgroup $\langle y\rangle$ of order q. Then if G is also an OP tiling group, then every inner automorphism of G has order 2 when restricted to $\langle y\rangle$.*

Proof. Since $\langle y\rangle$ is characteristic then we have

$$aya^{-1} = y^s,\ s^l \equiv 1 \bmod q, \tag{20}$$
$$byb^{-1} = y^t,\ t^m \equiv 1 \bmod q,$$
$$cyc^{-1} = y^u,\ u^n \equiv 1 \bmod q, \tag{21}$$
$$\theta(y) = y^w, w^2 \equiv 1 \bmod q.$$

The congruence relations hold since $\mathrm{Ad}_a, \mathrm{Ad}_b$, Ad_c and θ have orders dividing l, m, n and 2 respectively. Now

$$\theta(aya^{-1}) = \theta(a)\theta(y)\theta(a^{-1}) = a^{-1}y^w a = y^{ws'},\ \text{where } ss' \equiv 1 \bmod q,$$

and

$$\theta(aya^{-1}) = \theta(y^s) = y^{ws}.$$

Thus $ws \equiv ws' \bmod q$ and hence $s \equiv s' \bmod q$ since w is invertible $\bmod q$. As $ss' \equiv 1 \bmod q$, then $s^2 \equiv 1 \bmod q$. Similar arguments apply to b. Now the image of Ad_G in $\mathrm{Aut}(\langle y\rangle)$ is generated by elements of order dividing 2 since $G = \langle a, b\rangle$. But $\mathrm{Aut}(\langle y\rangle)$ is abelian, so every Ad_g, $g \in G$, has order dividing 2. ■

Remark 37 *The additional relations in 20 are often enough to conclude that G is abelian if G^* exists. This is certainly the case in Proposition 35 since G has odd order. On the other hand if any two of l, m, n are odd and G^* exists then $\langle y\rangle \subseteq Z(G)$ since G is generated by any two of a, b, c.*

Example 38 *Let $p > 5$ be a prime of the form $4s + 1$. There are two rotation groups with branching data $|G| = 4p$ and $(l, m, n) = (4, 4, p)$, acting on a surface of genus*

$$1 + \frac{4p}{2}(1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{p}) = p - 1,$$

such that one is a tiling group and the other is not. For, the p-Sylow subgroup is characteristic and there is complementary cyclic subgroup of order 4 because *of the specification of the branching data. The branching data also precludes an abelian G. The two possibilities for the groups are*

$$G_1 = D_{4,p,t} = \left\langle x, y : x^4 = y^p = 1, y^x = y^t \right\rangle,$$

where t is a primitive 4'th root of $1 \bmod p$ *and*

$$G_2 = D_{4,p,-1} = \left\langle x, y : x^4 = y^p = 1, y^x = y^{-1} \right\rangle.$$

Generating vectors for the groups may be taken as (x^{-1}, xy, y^{-1}) *in both cases. According to the last proposition only* G_2 *could be a tiling group and indeed* θ *is given by*

$$\theta(x) = x^{-1}, \theta(y) = y.$$

6 Rotation and tiling groups - genus 2-7

This section and the next contain tables of rotation and tiling groups found for surfaces of genus 2-13. The tables in this section cover only genus 2-7, but are fairly detailed, giving a group presentation, and a generating vector for each isometry class of actions. In the next section summary tables of the various actions as well as a complete list of all groups with the branching data, and the number of kaleidoscopic and non-kaleidoscopic vectors is given. The complete list of actions as well as the scripts used to construct them are available in electronic format at the website [17].

In the tables in this section there has been some effort to exhibit the various groups as standard constructions say, cyclic, abelian, split metacyclic, standard permutation groups or direct and semi-direct products of such. In the more general tables in the next section and the electronic lists the groups are only split into broad classes such as cyclic, abelian, solvable groups in PC format and non-solvable permutation groups. Presentations of the groups and generating vectors are only given on the website. The tables for genus 2 and 3 were originally computed by hand in [2], except for one omission, namely the group of order 48 with branching data (2,3,12) on a surface of genus 3. Moreover, there are the following errors in [2]: the group of order 21 in genus 3 should be a $(3,3,7)$-action, and the generators of $PSL_2(7)$ are given incorrectly. The genus 2 and 3 tables have therefore been included for completeness and correctness. For each group a list of inequivalent generating vectors is given. The tables give the order of the group, the (l,m,n)-triple for the group, the group or a presentation of the group, and whether or not the rotation group is also a tiling group. For polycyclic presentations the words "& abelian relations" indicate that, unless otherwise specified, generators commute. We use the following notation for groups, and the existence of a tiling:

- $\mathbb{Z}_n$: cyclic group of order n.
- $D_{p,q,r} = \mathbb{Z}_p \ltimes \mathbb{Z}_q = \langle x, y \mid x^p = y^q = 1, y^x = y^r \rangle$: split metacyclic group of order pq, note $r^p \equiv 1 \bmod q$. See section 5.

- Σ_n : symmetric group on n symbols.
- A_n : alternating group on n symbols.
- $GL_n(q)$, $SL_n(q)$, $PSL_n(q)$, general linear, special linear and, projective special linear group of $n \times n$ matrices over a field of q elements.
- If a semi-direct product $H \ltimes N$ is given a presentation then the first generators will represent H and followed by generators representing N. The action of H on N will be specified by conjugation relations.
- Some groups are specified as being generated by permutations.
- All groups are represented in multiplicative format.
- The last column indicates whether the full tiling group G^* exists and the tiling $\mathcal{T}$ is kaleidoscopic.

Table 6.1 Genus 2 Rotation and Tiling Groups

$\lvert G\rvert$	(l,m,n)	G	(a,b,c)	G^*
5	$(5,5,5)$	$\mathbb{Z}_5 = \langle x \mid x^5 = 1\rangle$	(x,x,x^3)	Yes
6	$(3,6,6)$	$\mathbb{Z}_6 = \langle x \mid x^6 = 1\rangle$	(x^4,x,x)	Yes
8	$(2,8,8)$	$\mathbb{Z}_8 = \langle x \mid x^8 = 1\rangle$	(x^4,x,x^3)	Yes
8	$(4,4,4)$	$\langle x,y \mid x^4 = y^4 = 1,$ $x^2 = y^2, y^x = y^{-1}\rangle$	(x,y,yx)	Yes
10	$(2,5,10)$	$\mathbb{Z}_{10} = \langle x \mid x^{10} = 1\rangle$	(x^5,x^2,x^3)	Yes
12	$(2,6,6)$	$\mathbb{Z}_6 \times \mathbb{Z}_2 = \langle x \mid x^6 = 1\rangle \times$ $\langle y \mid y^2 = 1\rangle$	(y,yx^{-1},x)	Yes
12	$(3,4,4)$	$D_{4,3,-1} = \langle x,y \mid x^4 = y^3 = 1,$ $y^x = y^{-1}\rangle$	$(y,(xy)^{-1},x)$	Yes
16	$(2,4,8)$	$D_{2,8,3} = \langle x,y \mid x^2 = y^8 = 1,$ $y^x = y^3\rangle$	$(x,(yx)^{-1},y)$	Yes
24	$(2,4,6)$	$\langle x,y,z,w \mid x^2 = y^2 = z^2 = 1,$ $w^3 = 1, z^x = zy, w^x = w^{-1},$ & abelian relations$\rangle$	$(x,(zwx)^{-1},zw)$	Yes
24	$(3,3,4)$	$SL_2(3) = \langle x,y \mid x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$ $y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\rangle$	$(x,(yx)^{-1},y)$	Yes
48	$(2,3,8)$	$GL_2(3) = \langle x,y \mid x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$ $y = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\rangle$	$(x,y,(xy)^{-1})$	Yes

Table 6.2.1 Genus 3 Rotation and Tiling Groups - part 1

$\lvert G\rvert$	(l,m,n)	G	(a,b,c)	G^*
7	$(7,7,7)$	$\mathbb{Z}_7 = \langle x \mid x^7 = 1\rangle$	(x,x,x^5) (x,x^2,x^4)	Yes Yes
8	$(4,8,8)$	$\mathbb{Z}_8 = \langle x \mid x^8 = 1\rangle$	(x^6,x,x) (x^2,x,x^5)	Yes Yes
9	$(3,9,9)$	$\mathbb{Z}_9 = \langle x \mid x^9 = 1\rangle$	(x^3,x^5,x)	Yes
12	$(2,12,12)$	$\mathbb{Z}_{12} = \langle x \mid x^{12} = 1\rangle$	(x^6,x^5,x)	Yes
12	$(3,4,12)$	$\mathbb{Z}_{12} = \langle x \mid x^{12} = 1\rangle$	(x^8,x^3,x)	Yes
12	$(4,4,6)$	$D_{4,3,-1} = \langle x,y \mid x^4 = y^3 = 1,$ $y^x = y^{-1}\rangle$	(x,xy^2,x^2y)	Yes
14	$(2,7,14)$	$\mathbb{Z}_{14} = \langle x \mid x^{14} = 1\rangle$	(x^7,x^6,x)	Yes

Table 6.2.2 Genus 3 Rotation and Tiling Groups - part 2

$\|G\|$	(l, m, n)	G	(a, b, c)	G^*
16	$(2, 8, 8)$	$\mathbb{Z}_8 \times \mathbb{Z}_2 = \langle x \mid x^8 = 1\rangle \times$ $\langle y \mid y^2 = 1\rangle$	(y, yx^{-1}, x)	Yes
16	$(2, 8, 8)$	$D_{2,8,5} = \langle x, y \mid x^2 = y^8 = 1,$ $y^x = y^5\rangle$	(x, xy^{-1}, y)	Yes
16	$(4, 4, 4)$	$\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle x \mid x^4 = 1\rangle \times$ $\langle y \mid y^4 = 1\rangle$	$(x, y, (xy)^{-1})$	Yes
16	$(4, 4, 4)$	$D_{4,4,-1} = \langle x, y \mid x^4 = y^4 = 1,$ $y^x = y^{-1}\rangle$	$(x, y, (xy)^{-1})$	Yes
21	$(3, 3, 7)$	$D_{3,7,2} = \langle x, y \mid x^3 = y^7 = 1,$ $y^x = y^2\rangle$	$(x, (yx)^{-1}, y)$	**NO**
24	$(2, 4, 12)$	$D_{2,12,5} = \langle x, y \mid x^2 = y^{12} = 1,$ $y^x = y^5\rangle$	$(x, (yx)^{-1}, y)$	Yes
24	$(2, 6, 6)$	$\mathbb{Z}_2 \times A_4 = \langle x \mid x^2 = 1\rangle \times$ $\langle y, z \mid y = (1, 2, 3),$ $z = (1, 2)(3, 4)\rangle$	$(z, yx, (zyx)^{-1})$	Yes
24	$(3, 3, 6)$	$SL_2(3) = \langle x, y \mid x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$ $y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\rangle$	$(x, x^y, (xx^y)^{-1})$	Yes
24	$(3, 4, 4)$	$\Sigma_4 = \langle x, y \mid x = (1, 2, 3, 4),$ $y = (1, 4, 2, 3)\rangle$	$((xy)^{-1}, x, y)$	Yes
32	$(2, 4, 8)$	$\langle x, y, z \mid x^2 = y^2 = z^8 = 1,$ $z^x = yz^3,$ & abelian relations$\rangle$	(x, xz, z^{-1})	Yes
32	$(2, 4, 8)$	$\langle x, y, z \mid x^2 = y^2 = z^8 = 1,$ $y^x = yz^4, z^x = yz^3, z^y = z^5\rangle$	(x, xz, z^{-1})	Yes
48	$(2, 3, 12)$	$\langle x, y, z, w, v \mid y^3 = v^2 = 1,$ $x^2 = z^2 = w^2 = v, z^y = w,$ $w^y = zw, w^z = wv,$ & abelian relations$\rangle$	(xz, yzw, xy^2zw)	Yes
48	$(2, 4, 6)$	$\mathbb{Z}_2 \times \Sigma_4 = \langle x \mid x^2 = 1\rangle \times$ $\langle y, z \mid y = (1, 2, 3, 4), z = (1, 2)\rangle$	(xz, y^{-1}, yzx)	Yes
48	$(3, 3, 4)$	$\mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2 = \langle x, y, z \mid$ $x^3 = y^4 = z^4 = 1, y^x = z,$ $z^x = (yz)^{-1}$ & abelian relations$\rangle$	$(x, (yx)^{-1}, y)$	Yes
96	$(2, 3, 8)$	$\Sigma_3 \ltimes (\mathbb{Z}_4)^2 = \langle x, y, z, w \mid$ $x^2 = y^3 = z^4 = w^4 = 1,$ $y^x = y^2, z^x = w, w^x = z, z^y = w,$ $w^y = (zw)^{-1},$ & abelian relations$\rangle$	(xy^{-1}, yw, xz^{-1})	Yes
168	$(2, 3, 7)$	$PSL_2(7) = \langle x, y \mid x = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}},$ $y = \overline{\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}}\rangle$	$(x, y, (xy)^{-1})$	Yes

Table 6.3.1 Genus 4 Rotation and Tiling Groups - part 1

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
9	$(9,9,9)$	$\mathbb{Z}_9 = \langle x \mid x^9 = 1\rangle$	(x,x,x^7)	Yes
10	$(5,10,10)$	$\mathbb{Z}_{10} = \langle x \mid x^{10} = 1\rangle$	(x^2,x,x^7) (x^8,x,x)	Yes Yes
12	$(3,12,12)$	$\mathbb{Z}_{12} = \langle x \mid x^{12} = 1\rangle$	(x^4,x,x^7)	Yes
12	$(4,6,12)$	$\mathbb{Z}_{12} = \langle x \mid x^{12} = 1\rangle$	(x^3,x^2,x^7)	Yes
12	$(6,6,6)$	$\mathbb{Z}_6 \times \mathbb{Z}_2 = \langle x \mid x^6 = 1\rangle \times$ $\langle y \mid y^2 = 1\rangle$	(x,xy,x^4y)	Yes
15	$(3,5,15)$	$\mathbb{Z}_{15} = \langle x \mid x^{15} = 1\rangle$	(x^5,x^3,x^7)	Yes
16	$(2,16,16)$	$\mathbb{Z}_{16} = \langle x \mid x^{16} = 1\rangle$	(x^8,x^3,x^5)	Yes
16	$(4,4,8)$	$\langle x,y,z,w \mid x^2 = y^2 = z^2 = w,$ $w^2 = 1, y^x = yz, z^x = zw,$ $z^y = zw,$ & abelian relations$\rangle$	(y,xzw,xyz)	Yes
18	$(2,9,18)$	$\mathbb{Z}_{18} = \langle x \mid x^{18} = 1\rangle$	(x^9,x^2,x^7)	Yes
18	$(3,6,6)$	$\mathbb{Z}_6 \times \mathbb{Z}_3 = \langle x \mid x^6 = 1\rangle \times$ $\langle y \mid y^3 = 1\rangle$	(y,xy,x^5y)	Yes
18	$(3,6,6)$	$\Sigma_3 \times \mathbb{Z}_3 = \langle x,y \mid x = (1,2,3),$ $y = (1,2)\rangle \times \langle z \mid z^3 = 1\rangle$	(xz,yz,yx^2z) (x,yz^2,yx^2z)	Yes Yes
20	$(2,10,10)$	$\mathbb{Z}_{10} \times \mathbb{Z}_2 = \langle x \mid x^{10} = 1\rangle \times$ $\langle y \mid y^2 = 1\rangle$	(y,xy,x^9)	Yes
20	$(4,4,5)$	$D_{4,5,-1} = \langle x,y \mid x^4 = y^5 = 1,$ $y^x = y^{-1}\rangle$	$(x,(yx)^{-1},y)$	Yes
20	$(4,4,5)$	$D_{4,5,2} = \langle x,y \mid x^4 = y^5 = 1,$ $y^x = y^2\rangle$	$(x,(yx)^{-1},y)$	**NO**
24	$(2,6,12)$	$D_{2,12,7} = \langle x,y \mid x^2 = y^{12} = 1,$ $y^x = y^7\rangle$	(x,xy^{-1},y)	Yes
24	$(3,4,6)$	$\langle x,y,z,w \mid x^3 = 1, y^2 = z^2 = w,$ $w^2 = 1, y^x = z, z^x = yz,$ $z^y = zw,$ & abelian relations$\rangle$	$(x,yw,(xyw)^{-1})$	Yes

Table 6.3.2 Genus 4 Rotation and Tiling Groups - part 2

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
32	$(2,4,16)$	$D_{2,16,7} = \langle x, y \mid x^2 = y^{16} = 1,$ $y^x = y^7\rangle$	(x, xy^{-1}, y)	Yes
36	$(2,6,6)$	$(\mathbb{Z}_2)^2 \ltimes (\mathbb{Z}_3)^2 = \langle x, y, z, w \mid$ $x^2 = y^2 = z^3 = w^3 = 1,$ $z^y = z^2, w^x = w^2,$ & abelian relations$\rangle$	(xy, yzw^2, xz^2w^2)	Yes
36	$(2,6,6)$	$(\mathbb{Z}_2)^2 \ltimes (\mathbb{Z}_3)^2 = \langle x, y, z, w \mid$ $x^2 = y^2 = z^3 = w^3 = 1,$ $w^x = w^2$, & abelian relations$\rangle$	(x, yzw^2, xyz^2w^2)	Yes
36	$(3,3,6)$	$(\mathbb{Z}_3)^2 \ltimes (\mathbb{Z}_2)^2 = \langle x, y, z, w \mid$ $x^3 = y^3 = z^2 = w^2 = 1,$ $z^x = w, w^x = zw,$ & abelian relations$\rangle$	(x, x^2yzw, y^2zw)	Yes
36	$(3,4,4)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2 = \langle x, y, z \mid$ $x^4 = y^3 = z^3 = 1,$ $y^x = yz^2, z^x = y^2z^2,$ & abelian relations$\rangle$	$(y, (xy)^{-1}, x)$	Yes
40	$(2,4,10)$	$\langle x, y, z, w \mid x^2 = y^2 = z^2 = 1,$ $w^5 = 1, y^x = yz, w^x = w^4,$ & abelian relations$\rangle$	$(x, xyzw, yzw^4)$	Yes
60	$(2,5,5)$	$A_5 = \langle x, y \mid x = (1,2)(3,4),$ $y = (1,2,3,4,5)\rangle$	(x, xy^2, y^3)	Yes
72	$(2,3,12)$	$\langle x, y, z, w, v \mid x^2 = y^3 = z^3 = 1,$ $w^2 = v^3 = 1, z^x = z^2, w^x = v,$ $w^z = v, v^x = w, v^z = wv,$ & abelian relations$\rangle$	(x, yz^2wv, xy^2z^2v)	Yes
72	$(2,4,6)$	$\langle x, y, z, w, v \mid x^2 = y^2 = z^2 = 1,$ $w^3 = v^3 = 1, y^x = yz, w^x = w^2,$ $w^y = v, w^z = w^2, v^y = w,$ $v^z = v^2$, & abelian relations$\rangle$	$(y, xyzw^2v^2, xw^2v)$	Yes
120	$(2,4,5)$	$\Sigma_5 = \langle x, y \mid x = (1,2),$ $y = (1,2,3,4,5)\rangle$	$(x, (yx)^{-1}, y)$	Yes

Table 6.4.1 Genus 5 Rotation and Tiling Groups - part 1

$\lvert G\rvert$	(l,m,n)	G	(a,b,c)	G^*
11	$(11,11,11)$	$\mathbb{Z}_{11}=\langle x \mid x^{11}=1\rangle$	(x,x,x^9) (x,x^2,x^8)	Yes Yes
12	$(6,12,12)$	$\mathbb{Z}_{12}=\langle x \mid x^{12}=1\rangle$	(x^2,x^5,x^5)	Yes
15	$(3,15,15)$	$\mathbb{Z}_{15}=\langle x \mid x^{15}=1\rangle$	(x^5,x^2,x^8)	Yes
16	$(4,8,8)$	$\mathbb{Z}_8\times\mathbb{Z}_2=\langle x \mid x^8=1\rangle\times$ $\langle y \mid y^2=1\rangle$	(x^2y,x,x^5y)	Yes
16	$(4,8,8)$	$D_{2,8,5}=\langle x,y \mid x^2=y^8=1,$ $y^x=y^5\rangle$	(xy^2,xy,y^5)	Yes
20	$(2,20,20)$	$\mathbb{Z}_{20}=\langle x \mid x^{20}=1\rangle$	(x^{10},x,x^9)	Yes
20	$(4,4,10)$	$D_{4,5,-1}=\langle x,y \mid x^4=y^5=1,$ $y^x=y^{-1}\rangle$	(x,xy,x^2y^4)	Yes
22	$(2,11,22)$	$\mathbb{Z}_{22}=\langle x \mid x^{22}=1\rangle$	(x^{11},x^2,x^9)	Yes
24	$(2,12,12)$	$\mathbb{Z}_{12}\times\mathbb{Z}_2=\langle x \mid x^{12}=1\rangle\times$ $\langle y \mid y^2=1\rangle$	(x^6y,x,x^5y)	Yes
24	$(3,6,6)$	$\mathbb{Z}_3\ltimes(\mathbb{Z}_2)^3=\langle x,y,z,w \mid x^3=1,$ $y^2=z^2=w^2=1,z^x=w,$ $w^x=zw,$ & abelian relations$\rangle$	$(x,xyzw,xyz)$	Yes
24	$(4,4,6)$	$D_{4,6,-1}=\langle x,y \mid x^4=y^6=1,$ $y^x=y^{-1}\rangle$	$(x,(yx)^{-1},y)$	Yes
30	$(2,6,15)$	$D_{6,5,-1}=\langle x,y \mid x^6=y^5=1,$ $y^x=y^{-1}\rangle$	(x^3,xy,x^2y^{-1})	Yes
32	$(2,8,8)$	$\mathbb{Z}_8\ltimes(\mathbb{Z}_2)^2=\langle x,y,z \mid x^8=1,$ $y^2=z^2=1,y^x=yz,$ & abelian relations$\rangle$	$(y,(xy)^{-1},x)$	Yes
32	$(2,8,8)$	$\langle x,y,z,w,v \mid x^2=w,w^2=v,$ $y^2=z^2=v^2=1,y^x=yz,$ $z^x=zv,w^y=wv,$ & abelian relations$\rangle$	$(y,xv,xywv)$	Yes
32	$(4,4,4)$	$\mathbb{Z}_4\ltimes(\mathbb{Z}_4\times\mathbb{Z}_2)=\langle x,y,z \mid$ $x^4=y^4=z^2=1,y^x=yz,$ & abelian relations$\rangle$	(y,xy^2,x^3y)	Yes
32	$(4,4,4)$	$\mathbb{Z}_4\ltimes(\mathbb{Z}_2)^3=\langle x,y,z,w \mid$ $x^4=y^2=z^2=w^2=1,$ $y^x=yz,z^x=zw,$ & abelian relations$\rangle$	$(x,xy,(x^2y)^{-1})$	Yes
40	$(2,4,20)$	$D_{2,20,9}=\langle x,y \mid x^2=y^{20}=1,$ $y^x=y^9\rangle$	(x,xy^{-1},y)	Yes
48	$(2,4,12)$	$\mathbb{Z}_2\ltimes(\mathbb{Z}_{12}\times\mathbb{Z}_2)=\langle x,y,z \mid$ $x^2=y^{12}=z^2=1,y^x=y^5z,$ & abelian relations$\rangle$	$(x,(yzx)^{-1},yz)$	Yes
48	$(3,4,4)$	$\langle x,y,z,w,v \mid x^2=y,y^2=1,$ $z^3=w^2=v^2=1,z^x=z^2,$ $w^x=v,w^z=v,v^x=w,$ $v^z=wv,$ & abelian relations$\rangle$	(z,xwv,xyz^2v) (z,xzw,xyv)	Yes Yes

Table 6.4.2 Genus 5 Rotation and Tiling Groups - part 2

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
60	$(3,3,5)$	$A_5 = \langle x, y \mid x = (1,2,3),$ $y = (3,4,5)\rangle$	$(x, y, (xy)^{-1})$	Yes
64	$(2,4,8)$	$\langle x, y, z, w, v, u \mid x^2 = w, z^2 = u,$ $y^2 = w^2 = v^2 = u^2 = 1,$ $y^x = yz, z^x = zv, z^y = zu,$ $w^y = wvu$ & abelian relations$\rangle$	$(y, xzu, xyzw)$	Yes
64	$(2,4,8)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4 = \langle x, y, z, w, v \mid$ $x^4 = y^2 = z^2 = w^2 = 1,$ $y^x = yz, z^x = zw, w^x = wv,$ & abelian relations$\rangle$	$(y, x, (yx)^{-1})$	Yes
80	$(2,5,5)$	$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4 = \langle x, y, z, w, v \mid$ $x^5 = y^2 = z^2 = w^2 = 1,$ $y^x = v, z^x = yv, w^x = yzv,$ $v^x = yzwv,$ & abelian relations$\rangle$	(y, xyw, x^4zwv)	Yes
96	$(2,4,6)$	$\Sigma_3 \ltimes (\mathbb{Z}_2)^4 = \langle x, y, z, w, v, u \mid$ $x^2 = y^3 = z^2 = w^2 = v^2 = 1,$ $u^2 = 1, y^x = y^2, z^x = zw,$ $v^x = u, u^x = v, v^y = u,$ $u^y = vu,$ & abelian relations$\rangle$	$(x, xyzv, (yzv)^{-1})$	Yes
96	$(3,3,4)$	$\langle x, y, z, w, v, u \mid x^3 = 1, y^2 = wv,$ $z^2 = w, w^2 = v^2 = u^2 = 1,$ $y^x = z, z^x = yzv, w^x = vu,$ $v^x = wvu, z^y = zu,$ & abelian relations$\rangle$	$(x, x^2zu, (zu)^{-1})$	Yes
120	$(2,3,10)$	$A_5 \times \mathbb{Z}_2 = \langle x, y \mid x = (1,2)(3,4),$ $y = (1,2,3,4,5)\rangle \times \langle z \mid z^2 = 1\rangle$	$(xz, (yx)^{-1}, yz)$	Yes
160	$(2,4,5)$	$D_5 \ltimes (\mathbb{Z}_2)^4 = \langle x, y, z, w, v, u \mid$ $x^2 = y^5 = z^2 = w^2 = v^2 = 1,$ $u^2 = 1, y^x = y^{-1}, z^x = zw,$ $z^y = zw, w^y = wv, v^x = zwu,$ $v^y = vu, u^x = zv, u^y = z,$ & abelian relations$\rangle$	(x, xy^4wv, ywv)	Yes
192	$(2,3,8)$	$\langle x, y, z, w, v, u, t \mid x^2 = y^3 = 1,$ $v^2 = u^2 = t^2 = 1, z^2 = vu,$ $w^2 = v, y^x = y^2, z^x = w,$ $z^y = wt, w^x = z, w^y = zwu,$ $w^z = wt, v^x = vu, v^y = ut,$ $u^y = vut,$ & abelian relations$\rangle$	$(x, yzvt, xyzwv)$	Yes

Table 6.5.1 Genus 6 Rotation and Tiling Groups - part 1

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
13	$(13,13,13)$	$\mathbb{Z}_{13} = \langle x \mid x^{13} = 1\rangle$	(x,x,x^{11}) (x,x^2,x^{10}) (x,x^3,x^9)	Yes Yes Yes
14	$(7,14,14)$	$\mathbb{Z}_{14} = \langle x \mid x^{14} = 1\rangle$	(x^{12},x,x) (x^{10},x,x^3) (x^2,x,x^{11})	Yes Yes Yes
15	$(5,15,15)$	$\mathbb{Z}_{15} = \langle x \mid x^{15} = 1\rangle$	(x^{12},x^2,x) (x^3,x,x^{11})	Yes Yes
16	$(4,16,16)$	$\mathbb{Z}_{16} = \langle x \mid x^{16} = 1\rangle$	(x^{12},x^3,x)	Yes
18	$(3,18,18)$	$\mathbb{Z}_{18} = \langle x \mid x^{18} = 1\rangle$	(x^6,x,x^{11})	Yes
20	$(4,5,20)$	$\mathbb{Z}_{20} = \langle x \mid x^{20} = 1\rangle$	(x^5,x^4,x^{11})	Yes
21	$(3,7,21)$	$\mathbb{Z}_{21} = \langle x \mid x^{21} = 1\rangle$	(x^7,x^3,x^{11})	Yes
24	$(2,24,24)$	$\mathbb{Z}_{24} = \langle x \mid x^{24} = 1\rangle$	(x^{12},x,x^{11})	Yes
24	$(3,8,8)$	$D_{8,3,-1} = \langle x,y \mid x^8 = y^3 = 1,$ $y^x = y^{-1}\rangle$	$(y,(xy)^{-1},x)$	Yes
24	$(4,4,12)$	$\langle x,y,z,w \mid x^2 = y^2 = z,$ $z^2 = w^3 = 1, y^x = yz,$ $w^x = w^2,$ & abelian relations$\rangle$	$(x,xyzw,yzw^2)$	Yes
24	$(4,6,6)$	$\langle x,y,z,w \mid x^3 = 1, y^2 = z^2 = w,$ $w^2 = 1, y^x = z,\ z^x = yz,$ $z^y = zw,$ & abelian relations$\rangle$	$(y,xyzw,x^2yz)$	Yes
24	$(4,6,6)$	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) =$ $\langle x,y,z,w,v \mid x^2 = y^2 = z^2 = 1,$ $w^3 = 1, y^x = yz,$ & abelian relations$\rangle$	(xyz,xzw^2,yzw)	Yes
25		$\mathbb{Z}_5 \times \mathbb{Z}_5 = \langle x \mid x^5 = 1\rangle\times$ $\langle y \mid y^5 = 1\rangle$	$(x,y,(xy)^{-1})$	Yes
26	$(2,13,26)$	$\mathbb{Z}_{26} = \langle x \mid x^{26} = 1\rangle$	(x^{13},x^2,x^{11})	Yes
28	$(2,14,14)$	$\mathbb{Z}_{14} \times \mathbb{Z}_2 = \langle x \mid x^{14} = 1\rangle\times$ $\langle y \mid y^2 = 1\rangle$	$(y,(xy)^{-1},x)$	Yes
28	$(4,4,7)$	$D_{4,7,-1} = \langle x,y \mid x^4 = y^7 = 1,$ $y^x = y^{-1}\rangle$	$(x,(yx)^{-1},y)$	Yes
30	$(2,10,15)$	$D_{2,15,11} = \langle x,y \mid x^2 = y^{15} = 1,$ $y^x = y^{11}\rangle$	$(x,(yx)^{-1},y)$	Yes

Table 6.5.2 Genus 6 Rotation and Tiling Groups part 2

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
36	$(2,9,9)$	$\mathbb{Z}_9 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle x,y,z \mid x^9 = y^2 = z^2 = 1, y^x = z, z^x = yz,$ & abelian relations$\rangle$	$(y,(xy)^{-1},x)$	Yes
39	$(3,3,13)$	$D_{3,13,3} = \langle x,y \mid x^3 = y^{13} = 1, y^x = y^3\rangle$	$(x,(yx)^{-1},y)$	**NO**
48	$(2,4,24)$	$D_{2,24,11} = \langle x,y \mid x^2 = y^{24} = 1, y^x = y^{11}\rangle$	$(x,(yx)^{-1},y)$	Yes
48	$(2,6,8)$	$\langle x,y,z,w,v \mid x^2 = y^2 = 1, w^2 = v^3 = 1,\ z^2 = w,\ y^x = yz, z^x = z^y = zw, v^x = v^2,$ & abelian relations$\rangle$	$(x,yv^2,xyzv^2)$	Yes
48	$(2,6,8)$	$\langle x,y,z,w,v \mid x^2 = y^3 = v^2 = 1, z^2 = w^2 = v,\ y^x = y^2,\ z^x = w, z^y = wv, w^x = z, w^y = zw, w^z = wv,$ & abelian relations$\rangle$	(x,y^2zwv,xy^2wv)	Yes
50	$(2,5,10)$	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_5 \times \mathbb{Z}_5) = \langle x,y,z \mid x^2 = y^5 = z^5 = 1, z^x = z^4,$ & abelian relations$\rangle$	(x,y^3z^3,xy^2z^3)	Yes
56	$(2,4,14)$	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7) = \langle x,y,z,w \mid x^2 = y^2 = z^2 = w^7 = 1, y^x = yz, w^x = w^6,$ & abelian relations$\rangle$	(x,xyw^2,yw^5)	Yes
72	$(2,4,9)$	$D_9 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle x,y,z,w \mid x^2 = y^9 = z^2 = w^2 = 1,\ y^x = y^{-1}, z^x = w,\ w^x = z, z^y = w, w^y = zw,$ & abelian relations$\rangle$	$(x,(yzx)^{-1},yz)$	Yes
75	$(3,3,5)$	$\mathbb{Z}_3 \ltimes (\mathbb{Z}_5 \times \mathbb{Z}_5) = \langle x,y,z \mid x^3 = y^5 = z^5 = 1,\ y^x = yz^3, z^x = y^4z^3,$ & abelian relations$\rangle$	(x,x^2y^4,y)	Yes
120	$(2,4,6)$	$\Sigma_5 = \langle x,y : x = (1,2)(3,4), y = (2,4,5,3)\rangle$	$(x,y,(xy)^{-1})$	Yes
150	$(2,3,10)$	$\Sigma_3 \ltimes (\mathbb{Z}_5 \times \mathbb{Z}_5) = \langle x,y,z,w : x^2 = y^3 = z^5 = w^5 = 1, y^x = y^2, z^y = zw^3,\ w^x = z^4w^4, w^y = z^4w^3,$ & abelian relations$\rangle$	$(x,y^2z^4w^3,xy^2w)$	Yes

Table 6.6.1 Genus 7 Rotation and Tiling Groups - part 1

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
15	$(15,15,15)$	$\mathbb{Z}_{15} = \langle x \mid x^{15}=1\rangle$	(x,x,x^{13})	Yes
16	$(8,16,16)$	$\mathbb{Z}_{16} = \langle x \mid x^{16}=1\rangle$	(x^2,x,x^{13}) (x^2,x^3,x^{11}) (x^2,x^7,x^7)	Yes Yes Yes
18	$(6,9,18)$	$\mathbb{Z}_{18} = \langle x \mid x^{18}=1\rangle$	(x^3,x^{14},x) (x^3,x^{10},x^5)	Yes Yes
20	$(4,10,20)$	$\mathbb{Z}_{20} = \langle x \mid x^{20}=1\rangle$	(x^5,x^{14},x)	Yes
21	$(3,21,21)$	$\mathbb{Z}_{21} = \langle x \mid x^{21}=1\rangle$	(x^7,x^{13},x)	Yes
24	$(3,8,24)$	$\mathbb{Z}_{24} = \langle x \mid x^{24}=1\rangle$	(x^8,x^{15},x)	Yes
24	$(4,6,12)$	$\mathbb{Z}_{12}\times\mathbb{Z}_2 = \langle x \mid x^{12}=1\rangle\times$ $\langle y \mid y^2=1\rangle$	(x^3,x^4y,x^5y)	Yes
24	$(6,6,6)$	$SL_2(3) = \langle x,y \mid x=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),$ $y=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\rangle$	$(xy^2,yxy,$ $(xy^3xy)^{-1})$	Yes
27	$(3,9,9)$	$\mathbb{Z}_9\times\mathbb{Z}_3 = \langle x \mid x^9=1\rangle\times$ $\langle y \mid y^3=1\rangle$	(y,x,x^8y^2)	Yes
27	$(3,9,9)$	$\langle x,y,z \mid x^3=z, y^3=z^3=1,$ $y^x=yz,$ & abelian relations$\rangle$	$(y,x,x^2y^2z^2)$	**NO**
28	$(2,28,28)$	$\mathbb{Z}_{28} = \langle x \mid x^{28}=1\rangle$	(x^{14},x^{13},x)	Yes
28	$(4,4,14)$	$D_{4,7,-1} = \langle x,y \mid x^4=y^7=1,$ $y^x=y^{-1}\rangle$	(x,xy,x^2y^{-1})	Yes
30	$(2,15,30)$	$\mathbb{Z}_{30} = \langle x \mid x^{30}=1\rangle$	(x^{15},x^{14},x)	Yes
32	$(2,16,16)$	$\mathbb{Z}_{16}\times\mathbb{Z}_2 = \langle x \mid x^{16}=1\rangle\times$ $\langle y \mid y^2=1\rangle$	$(y,x,x^{15}y)$	Yes
32	$(2,16,16)$	$D_{2,16,9} = \langle x,y \mid x^2=1,$ $y^{16}=1, y^x=y^9,$ & abelian relations$\rangle$	$(x,(yx)^{-1},y)$	Yes
32	$(4,4,8)$	$\langle x,y,z,w,v \mid x^2=w,$ $y^2=z^2=v,\ w^2=1,$ $v^2=1, y^x=yz, z^x=z^y=zv,$ & abelian relations$\rangle$	(y,xw,xyv)	Yes
32	$(4,4,8)$	$\langle x,y,z,w,v \mid x^2=w,$ $y^2=v^2=1, z^2=w^2=v,$ $y^x=yz, z^x=z^y=zv,$ & abelian relations$\rangle$	(xy,yw,xv)	Yes
32	$(4,4,8)$	$\langle x,y,z,w,v \mid x^2=w,\ y^2=z,$ $z^2=v, w^2=v^2=1,\ y^x=yz,$ $z^x=zv,$ & abelian relations$\rangle$	(x,xyw,yzv)	Yes
32	$(4,4,8)$	$\langle x,y,z,w,v \mid x^2=w,\ y^2=zv,$ $z^2=v, w^2=v^2=1,\ y^x=yz,$ $z^x=zv,$ & abelian relations$\rangle$	(x,xyw,yz)	Yes

Table 6.6.2 Genus 7 Rotation and Tiling Groups - part 2

$\|G\|$	(l,m,n)	G	(a,b,c)	G^*
36	$(3,4,12)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle x,y,z \mid x^4 = y^3 = z^3 = 1, y^x = y^2,$ & abelian relations$\rangle$	$(zy, xy, x^{-1}z^{-1})$	Yes
42	$(2,6,21)$	$D_{2,21,13} = \langle x,y \mid x^2 = y^{21} = 1, y^x = y^{13}\rangle$	$(x,(yx)^{-1},y)$	Yes
48	$(2,6,12)$	$\langle x,y,z,w,v \mid y^3 = v^2 = 1, x^2 = z^2 = w^2 = v,\ z^y = w, w^y = zw,\ w^z = wv,$ & abelian relations$\rangle$	$(xz, yzwv, xy^2zwv)$	Yes
48	$(3,4,6)$	$\langle x,y,z,w,v \mid x^2 = y^3 = v^2 = 1, z^2 = w^2 = v,\ z^y = w,\ w^y = zw, w^z = wv,$ & abelian relations$\rangle$	(y, xwv, xy^2z)	Yes
54	$(2,6,9)$	$D_{6,9,-1} = \langle x,y \mid x^6 = y^9 = 1, y^x = y^{-1}\rangle$	$(x^3, xy, y^{-1}x^2)$	Yes
54	$(2,6,9)$	$D_{6,9,2} = \langle x,y \mid x^6 = y^9 = 1, y^x = y^2\rangle$	$(x^3, xy, y^{-1}x^2)$	**NO**
56	$(2,4,28)$	$D_{2,28,13} = \langle x,y \mid x^2 = y^{28} = 1, y^x = y^{13}\rangle$	(x, xy, y^{-1})	Yes
56	$(2,7,7)$	$\mathbb{Z}_7 \ltimes (\mathbb{Z}_2)^3 = \langle x,y,z,w : x^7 = 1, y^2 = z^2 = w^2 = 1, y^x = z, z^x = w, w^x = yw,$ & abelian relations$\rangle$	$(y,(xy)^{-1},x)$	**NO**
64	$(2,4,16)$	$\langle x,y,z,w,v,u \mid x^2 = w,\ z^2 = vu, v^2 = u, y^2 = w^2 = u^2 = 1, y^x = yz, z^x = z^y = zv, v^x = vu, v^y = vu,$ & abelian relations$\rangle$	(y, x, xyw)	Yes
64	$(2,4,16)$	$\langle x,y,z,w,v,u \mid x^2 = w,\ z^2 = v, v^2 = u, y^2 = w^2 = u^2 = 1, y^x = yz, z^x = zv,\ z^y = zvu, w^y = wu, v^x = v^y = vu,$ & abelian relations$\rangle$	$(y, x, xywu)$	Yes
72	$(3,3,6)$	$\langle x,y,z,w,v \mid x^3 = y^3 = v^2 = 1, z^2 = w^2 = v,\ z^x = w,\ w^x = zw, w^z = wv,$ & abelian relations$\rangle$	(x, xyw, xy^2zwv)	Yes
144	$(2,3,12)$	$\langle x,y,z,w,v,u \mid x^2 = z^2 = w^2 = u, y^3 = v^3 = u^2 = 1, z^y = zw, w^y = z,\ w^z = wu, v^x = v^2,$ & abelian relations$\rangle$	(xz, y^2wv, xyv)	Yes
504	$(2,3,7)$	$PSL_2(8) = \langle x,y \mid x = (1,8)(2,3)(5,6)(7,9), y = (1,8,5)(2,3,7)(4,9,6)\rangle$	$(x,y,(xy)^{-1})$	Yes

7 Summary tables - genus 2-13

7.1 Summary Tables

Table 7.1 shows the number of actions in each genus broken up into five mutually exclusive classes: cyclic groups; non-cyclic Abelian groups; non-Abelian p-groups; non-Abelian solvable groups which are not p-groups; and non-solvable groups. Table 7.2 shows aggregated information from Table 7.1

Table 7.1 Number of isometrically inequivalent actions

Genus	Cyclic	Abelian NC*	p-group NA*	solvable NA*, NP*	non-solvable
2	4	1	2	4	0
3	8	2	4	10	1
4	8	3	2	13	2
5	6	2	7	13	2
6	14	2	0	16	1
7	11	3	8	12	1
8	10	2	2	15	4
9	17	3	18	32	4
10	18	4	3	45	3
11	10	2	5	27	3
12	25	3	0	28	0
13	13	6	16	29	4
Total	144	33	67	244	25

* NC = non-cyclic, NA = non-Abelian, NP = not a p-group.

Table 7.2 Number of isometrically inequivalent actions

Genus	abelian	solvable	non-solvable	total groups
2	5	11	0	11
3	10	24	1	25
4	11	26	2	28
5	8	28	2	30
6	16	32	1	33
7	14	34	1	35
8	12	29	4	33
9	20	70	4	74
10	22	70	3	73
11	12	44	3	47
12	28	56	0	56
13	19	64	4	68
total	177	488	25	513

It is interesting that there are isometrically inequivalent actions that have the same group and branching data. The corresponding tilings of these surfaces (if they existed) could not be distinguished by crude measures such as the angle of the triangles, the total number of triangles, or even the isomorphism class of the tiling group. In genus 14 there are three such inequivalent actions of $PSL_2(13)$, and is part of an infinite family of examples. In [13] the surfaces are distinguished geometrically by means of systoles. Ways of distinguishing the tilings combinatorially have not been explored. There is only one case of a group ($\sigma = 10, |G| = 216$) with a kaleidoscopic action and a non-kaleidoscopic with the same branching data. In Table 7.3 we record the frequency of occurrence of multiple actions for all groups, again broken down by structural group type. Not surprisingly, that multiple actions is most common for cyclic groups.

Table 7.3 Multiple Actions of Groups with the same Branching Data

# Actions	Cyclic	Abelian NC*	p-group NA*	solvable NA*, NP*	non-solvable
1	63	29	55	209	19
2	16	2	6	16	3
3	10			1	
4	2				
5	1				
6	1				
Total	144	33	67	244	25

* NC = non-cyclic, NA = non-Abelian, NP = not a p-group.

One of the motivations for this study was to classify tilings. The classification of the rotation groups shows that most rotation groups – indeed more than 92% – are derived from tilings. Table 7. 4 shows the non-kaleidoscopic rotation groups. Though there are no non-kaleidoscopic tilings for non-solvable groups up to genus 13, Singerman [10] reports a non-kaleidoscopic Hurwitz surface in genus 17.

Table 7. 4 Non-kaleidoscopic Actions

Genus	Abelian	p-group NA*	solvable NA*, NP*	non-solvable	Total
2	0	0	0	0	0
3	0	0	1	0	1
4	0	0	1	0	1
5	0	0	0	0	0
6	0	0	1	0	1
7	0	1	2	0	3
8	0	0	3	0	3
9	0	0	2	0	2
10	0	1	8	0	9
11	0	0	11	0	11
12	0	0	9	0	9
13	0	0	1	0	1
Total	0	2	39	0	41

* NC = non-cyclic, NA = non-Abelian, NP = not a p-group.

7.2 Tables of actions, genus 2-13

The tables that follow give for each triple of a genus σ a group G and branching triple (l, m, n) the number of isometric equivalence classes of actions separated into kaleidoscopic and non-kaleidoscopic classes. Presentations of all groups and the corresponding generating vectors for each action are available in electronic form at the website [17].

Notation

σ	genus of the surface
$\|G\|$	order of the rotation group G
Group	Z_n : cyclic group of order n
	$G(g, n)$: n'th small group of order g in the Magma data base
#Kal	number of kaleidoscopic actions yielding a tiling
	and, hence, a tiling group G^*
#non-Kal	number of non-kaleidoscopic actions
total	#Kal + #non-Kal
Type	C = cyclic group, A2 = 2 generator, non-cylic abelian group
	p-NA = non-abelian p-group
	S-NA-NP = non-abelian solvable group, but not a p-group
	NS = non-solvable

Table 7.5.1 Rotation and Tiling Groups, Genus 2-13 - part 1

σ	$\lvert G \rvert$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
2	5	$(5,5,5)$	Z_5	1	0	1	C
2	6	$(3,6,6)$	Z_6	1	0	1	C
2	8	$(2,8,8)$	Z_8	1	0	1	C
2	8	$(4,4,4)$	$G(8,4)$	1	0	1	p-NA
2	10	$(2,5,10)$	Z_{10}	1	0	1	C
2	12	$(2,6,6)$	$Z_2 \times Z_6$	1	0	1	A2
2	12	$(3,4,4)$	$G(12,1)$	1	0	1	S-NA-NP
2	16	$(2,4,8)$	$G(16,8)$	1	0	1	p-NA
2	24	$(2,4,6)$	$G(24,8)$	1	0	1	S-NA-NP
2	24	$(3,3,4)$	$G(24,3)$	1	0	1	S-NA-NP
2	48	$(2,3,8)$	$G(48,29)$	1	0	1	S-NA-NP
3	7	$(7,7,7)$	Z_7	2	0	2	C
3	8	$(4,8,8)$	Z_8	2	0	2	C
3	9	$(3,9,9)$	Z_9	1	0	1	C
3	12	$(2,12,12)$	Z_{12}	1	0	1	C
3	12	$(3,4,12)$	Z_{12}	1	0	1	C
3	12	$(4,4,6)$	$G(12,1)$	1	0	1	S-NA-NP
3	14	$(2,7,14)$	Z_{14}	1	0	1	C
3	16	$(2,8,8)$	$Z_2 \times Z_8$	1	0	1	A2
3	16	$(2,8,8)$	$G(16,6)$	1	0	1	p-NA
3	16	$(4,4,4)$	$Z_4 \times Z_4$	1	0	1	A2
3	16	$(4,4,4)$	$G(16,4)$	1	0	1	p-NA
3	21	$(3,3,7)$	$G(21,1)$	0	1	1	S-NA-NP
3	24	$(2,4,12)$	$G(24,5)$	1	0	1	S-NA-NP
3	24	$(2,6,6)$	$G(24,13)$	1	0	1	S-NA-NP
3	24	$(3,3,6)$	$G(24,3)$	1	0	1	S-NA-NP
3	24	$(3,4,4)$	$G(24,12)$	1	0	1	S-NA-NP
3	32	$(2,4,8)$	$G(32,9)$	1	0	1	p-NA
3	32	$(2,4,8)$	$G(32,11)$	1	0	1	p-NA
3	48	$(2,3,12)$	$G(48,33)$	1	0	1	S-NA-NP
3	48	$(2,4,6)$	$G(48,48)$	1	0	1	S-NA-NP
3	48	$(3,3,4)$	$G(48,3)$	1	0	1	S-NA-NP
3	96	$(2,3,8)$	$G(96,64)$	1	0	1	S-NA-NP
3	168	$(2,3,7)$	$G(168,42)$	1	0	1	NS

Table 7.5.2 Rotation and Tiling Groups, Genus 2-13 - part 2

σ	$\|G\|$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
4	9	$(9,9,9)$	Z_9	1	0	1	C
4	10	$(5,10,10)$	Z_{10}	2	0	2	C
4	12	$(3,12,12)$	Z_{12}	1	0	1	C
4	12	$(4,6,12)$	Z_{12}	1	0	1	C
4	12	$(6,6,6)$	$Z_2 \times Z_6$	1	0	1	A2
4	15	$(3,5,15)$	Z_{15}	1	0	1	C
4	16	$(2,16,16)$	Z_{16}	1	0	1	C
4	16	$(4,4,8)$	$G(16,9)$	1	0	1	p-NA
4	18	$(2,9,18)$	Z_{18}	1	0	1	C
4	18	$(3,6,6)$	$Z_3 \times Z_6$	1	0	1	A2
4	18	$(3,6,6)$	$G(18,3)$	2	0	2	S-NA-NP
4	20	$(2,10,10)$	$Z_2 \times Z_{10}$	1	0	1	A2
4	20	$(4,4,5)$	$G(20,1)$	1	0	1	S-NA-NP
4	20	$(4,4,5)$	$G(20,3)$	0	1	1	S-NA-NP
4	24	$(2,6,12)$	$G(24,10)$	1	0	1	S-NA-NP
4	24	$(3,4,6)$	$G(24,3)$	1	0	1	S-NA-NP
4	32	$(2,4,16)$	$G(32,19)$	1	0	1	p-NA
4	36	$(2,6,6)$	$G(36,10)$	1	0	1	S-NA-NP
4	36	$(2,6,6)$	$G(36,12)$	1	0	1	S-NA-NP
4	36	$(3,3,6)$	$G(36,11)$	1	0	1	S-NA-NP
4	36	$(3,4,4)$	$G(36,9)$	1	0	1	S-NA-NP
4	40	$(2,4,10)$	$G(40,8)$	1	0	1	S-NA-NP
4	60	$(2,5,5)$	$G(60,5)$	1	0	1	NS
4	72	$(2,3,12)$	$G(72,42)$	1	0	1	S-NA-NP
4	72	$(2,4,6)$	$G(72,40)$	1	0	1	S-NA-NP
4	120	$(2,4,5)$	$G(120,34)$	1	0	1	NS
5	11	$(11,11,11)$	Z_{11}	2	0	2	C
5	12	$(6,12,12)$	Z_{12}	1	0	1	C
5	15	$(3,15,15)$	Z_{15}	1	0	1	C
5	16	$(4,8,8)$	$Z_2 \times Z_8$	1	0	1	A2
5	16	$(4,8,8)$	$G(16,6)$	1	0	1	p-NA
5	20	$(2,20,20)$	Z_{20}	1	0	1	C
5	20	$(4,4,10)$	$G(20,1)$	1	0	1	S-NA-NP
5	22	$(2,11,22)$	Z_{22}	1	0	1	C

Table 7.5.3 Rotation and Tiling Groups, Genus 2-13 - part 3

σ	$\|G\|$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
5	24	$(2,12,12)$	$Z_2 \times Z_{12}$	1	0	1	A2
5	24	$(3,6,6)$	$G(24,13)$	1	0	1	S-NA-NP
5	24	$(4,4,6)$	$G(24,7)$	1	0	1	S-NA-NP
5	30	$(2,6,15)$	$G(30,2)$	1	0	1	S-NA-NP
5	32	$(2,8,8)$	$G(32,5)$	1	0	1	p-NA
5	32	$(2,8,8)$	$G(32,7)$	1	0	1	p-NA
5	32	$(4,4,4)$	$G(32,2)$	1	0	1	p-NA
5	32	$(4,4,4)$	$G(32,6)$	1	0	1	p-NA
5	40	$(2,4,20)$	$G(40,5)$	1	0	1	S-NA-NP
5	48	$(2,4,12)$	$G(48,14)$	1	0	1	S-NA-NP
5	48	$(3,4,4)$	$G(48,30)$	2	0	2	S-NA-NP
5	60	$(3,3,5)$	$G(60,5)$	1	0	1	NS
5	64	$(2,4,8)$	$G(64,8)$	1	0	1	p-NA
5	64	$(2,4,8)$	$G(64,32)$	1	0	1	p-NA
5	80	$(2,5,5)$	$G(80,49)$	1	0	1	S-NA-NP
5	96	$(2,4,6)$	$G(96,195)$	1	0	1	S-NA-NP
5	96	$(3,3,4)$	$G(96,3)$	1	0	1	S-NA-NP
5	120	$(2,3,10)$	$G(120,35)$	1	0	1	NS
5	160	$(2,4,5)$	$G(160,234)$	1	0	1	S-NA-NP
5	192	$(2,3,8)$	$G(192,181)$	1	0	1	S-NA-NP
6	13	$(13,13,13)$	Z_{13}	3	0	3	C
6	14	$(7,14,14)$	Z_{14}	3	0	3	C
6	15	$(5,15,15)$	Z_{15}	2	0	2	C
6	16	$(4,16,16)$	Z_{16}	1	0	1	C
6	18	$(3,18,18)$	Z_{18}	1	0	1	C
6	20	$(4,5,20)$	Z_{20}	1	0	1	C
6	21	$(3,7,21)$	Z_{21}	1	0	1	C
6	24	$(2,24,24)$	Z_{24}	1	0	1	C
6	24	$(3,8,8)$	$G(24,1)$	1	0	1	S-NA-NP
6	24	$(4,4,12)$	$G(24,4)$	1	0	1	S-NA-NP
6	24	$(4,6,6)$	$G(24,3)$	1	0	1	S-NA-NP
6	24	$(4,6,6)$	$G(24,10)$	1	0	1	S-NA-NP
6	25	$(5,5,5)$	$Z_5 \times Z_5$	1	0	1	A2
6	26	$(2,13,26)$	Z_{26}	1	0	1	C

Table 7.5.4 Rotation and Tiling Groups, Genus 2-13 - part 4

σ	$\lvert G\rvert$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
6	28	$(2,14,14)$	$Z_2\times Z_{14}$	1	0	1	A2
6	28	$(4,4,7)$	$G(28,1)$	1	0	1	S-NA-NP
6	30	$(2,10,15)$	$G(30,1)$	1	0	1	S-NA-NP
6	36	$(2,9,9)$	$G(36,3)$	1	0	1	S-NA-NP
6	39	$(3,3,13)$	$G(39,1)$	0	1	1	S-NA-NP
6	48	$(2,4,24)$	$G(48,6)$	1	0	1	S-NA-NP
6	48	$(2,6,8)$	$G(48,15)$	1	0	1	S-NA-NP
6	48	$(2,6,8)$	$G(48,29)$	1	0	1	S-NA-NP
6	50	$(2,5,10)$	$G(50,3)$	1	0	1	S-NA-NP
6	56	$(2,4,14)$	$G(56,7)$	1	0	1	S-NA-NP
6	72	$(2,4,9)$	$G(72,15)$	1	0	1	S-NA-NP
6	75	$(3,3,5)$	$G(75,2)$	1	0	1	S-NA-NP
6	120	$(2,4,6)$	$G(120,34)$	1	0	1	NS
6	150	$(2,3,10)$	$G(150,5)$	1	0	1	S-NA-NP
7	15	$(15,15,15)$	Z_{15}	1	0	1	C
7	16	$(8,16,16)$	Z_{16}	3	0	3	C
7	18	$(6,9,18)$	Z_{18}	2	0	2	C
7	20	$(4,10,20)$	Z_{20}	1	0	1	C
7	21	$(3,21,21)$	Z_{21}	1	0	1	C
7	24	$(3,8,24)$	Z_{24}	1	0	1	C
7	24	$(4,6,12)$	$Z_2\times Z_{12}$	1	0	1	A2
7	24	$(6,6,6)$	$G(24,3)$	1	0	1	S-NA-NP
7	27	$(3,9,9)$	$Z_3\times Z_9$	1	0	1	A2
7	27	$(3,9,9)$	$G(27,4)$	0	1	1	p-NA
7	28	$(2,28,28)$	Z_{28}	1	0	1	C
7	28	$(4,4,14)$	$G(28,1)$	1	0	1	S-NA-NP
7	30	$(2,15,30)$	Z_{30}	1	0	1	C
7	32	$(2,16,16)$	$Z_2\times Z_{16}$	1	0	1	A2
7	32	$(2,16,16)$	$G(32,17)$	1	0	1	p-NA
7	32	$(4,4,8)$	$G(32,10)$	1	0	1	p-NA
7	32	$(4,4,8)$	$G(32,11)$	1	0	1	p-NA
7	32	$(4,4,8)$	$G(32,13)$	1	0	1	p-NA
7	32	$(4,4,8)$	$G(32,14)$	1	0	1	p-NA
7	36	$(3,4,12)$	$G(36,6)$	1	0	1	S-NA-NP

Table 7.5.5 Rotation and Tiling Groups, Genus 2-13 - part 5

σ	$\|G\|$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
7	42	$(2,6,21)$	$G(42,4)$	1	0	1	S-NA-NP
7	48	$(2,6,12)$	$G(48,33)$	1	0	1	S-NA-NP
7	48	$(3,4,6)$	$G(48,32)$	1	0	1	S-NA-NP
7	54	$(2,6,9)$	$G(54,3)$	1	0	1	S-NA-NP
7	54	$(2,6,9)$	$G(54,6)$	0	1	1	S-NA-NP
7	56	$(2,4,28)$	$G(56,4)$	1	0	1	S-NA-NP
7	56	$(2,7,7)$	$G(56,11)$	0	1	1	S-NA-NP
7	64	$(2,4,16)$	$G(64,38)$	1	0	1	p-NA
7	64	$(2,4,16)$	$G(64,41)$	1	0	1	p-NA
7	72	$(3,3,6)$	$G(72,25)$	1	0	1	S-NA-NP
7	144	$(2,3,12)$	$G(144,127)$	1	0	1	S-NA-NP
7	504	$(2,3,7)$	$G(504,156)$	1	0	1	NS
8	17	$(17,17,17)$	Z_{17}	3	0	3	C
8	18	$(9,18,18)$	Z_{18}	2	0	2	C
8	20	$(10,10,10)$	$Z_2 \times Z_{10}$	1	0	1	A2
8	20	$(5,20,20)$	Z_{20}	2	0	2	C
8	24	$(3,24,24)$	Z_{24}	1	0	1	C
8	24	$(4,12,12)$	$G(24,11)$	1	0	1	S-NA-NP
8	24	$(6,6,12)$	$G(24,10)$	1	0	1	S-NA-NP
8	24	$(6,8,8)$	$G(24,1)$	1	0	1	S-NA-NP
8	30	$(3,10,10)$	$G(30,1)$	1	0	1	S-NA-NP
8	30	$(5,6,6)$	$G(30,2)$	1	0	1	S-NA-NP
8	32	$(2,32,32)$	Z_{32}	1	0	1	C
8	32	$(4,4,16)$	$G(32,20)$	1	0	1	p-NA
8	34	$(2,17,34)$	Z_{34}	1	0	1	C
8	36	$(2,18,18)$	$Z_2 \times Z_{18}$	1	0	1	A2
8	36	$(4,4,9)$	$G(36,1)$	1	0	1	S-NA-NP
8	40	$(2,10,20)$	$G(40,10)$	1	0	1	S-NA-NP
8	42	$(3,6,6)$	$G(42,1)$	0	1	1	S-NA-NP
8	42	$(3,6,6)$	$G(42,2)$	0	1	1	S-NA-NP
8	48	$(2,6,24)$	$G(48,25)$	1	0	1	S-NA-NP
8	48	$(2,8,12)$	$G(48,17)$	1	0	1	S-NA-NP
8	48	$(3,4,8)$	$G(48,28)$	1	0	1	S-NA-NP
8	60	$(2,6,10)$	$G(60,8)$	1	0	1	S-NA-NP

Table 7.5.6 Rotation and Tiling Groups, Genus 2-13 - part 6

σ	$\lvert G\rvert$	(l, m, n)	Group	#Kal	#non-Kal	total	Type
8	64	$(2, 4, 32)$	$G(64, 53)$	1	0	1	p-NA
8	72	$(2, 4, 18)$	$G(72, 8)$	1	0	1	S-NA-NP
8	84	$(2, 6, 6)$	$G(84, 7)$	0	1	1	S-NA-NP
8	168	$(3, 3, 4)$	$G(168, 42)$	2	0	2	NS
8	336	$(2, 3, 8)$	$G(336, 208)$	2	0	2	NS
9	19	$(19, 19, 19)$	Z_{19}	4	0	4	C
9	20	$(10, 20, 20)$	Z_{20}	2	0	2	C
9	21	$(7, 21, 21)$	Z_{21}	3	0	3	C
9	24	$(4, 24, 24)$	Z_{24}	2	0	2	C
9	24	$(6, 12, 12)$	$Z_2 \times Z_{12}$	1	0	1	A2
9	24	$(6, 8, 24)$	Z_{24}	1	0	1	C
9	24	$(8, 8, 12)$	$G(24, 1)$	2	0	2	S-NA-NP
9	27	$(3, 27, 27)$	Z_{27}	1	0	1	C
9	28	$(4, 7, 28)$	Z_{28}	1	0	1	C
9	30	$(3, 10, 30)$	Z_{30}	1	0	1	C
9	32	$(4, 8, 8)$	$Z_4 \times Z_8$	1	0	1	A2
9	32	$(4, 8, 8)$	$G(32, 4)$	1	0	1	p-NA
9	32	$(4, 8, 8)$	$G(32, 5)$	1	0	1	p-NA
9	32	$(4, 8, 8)$	$G(32, 8)$	1	0	1	p-NA
9	32	$(4, 8, 8)$	$G(32, 12)$	2	0	2	p-NA
9	36	$(2, 36, 36)$	Z_{36}	1	0	1	C
9	36	$(4, 4, 18)$	$G(36, 1)$	1	0	1	S-NA-NP
9	38	$(2, 19, 38)$	Z_{38}	1	0	1	C
9	40	$(2, 20, 20)$	$Z_2 \times Z_{20}$	1	0	1	A2
9	40	$(4, 4, 10)$	$G(40, 7)$	1	0	1	S-NA-NP
9	40	$(4, 4, 10)$	$G(40, 12)$	0	1	1	S-NA-NP
9	42	$(2, 14, 21)$	$G(42, 3)$	1	0	1	S-NA-NP
9	48	$(2, 12, 12)$	$G(48, 21)$	1	0	1	S-NA-NP
9	48	$(2, 12, 12)$	$G(48, 31)$	2	0	2	S-NA-NP
9	48	$(2, 8, 24)$	$G(48, 4)$	1	0	1	S-NA-NP
9	48	$(2, 8, 24)$	$G(48, 5)$	1	0	1	S-NA-NP
9	48	$(3, 4, 12)$	$G(48, 31)$	1	0	1	S-NA-NP
9	48	$(3, 6, 6)$	$G(48, 32)$	1	0	1	S-NA-NP
9	48	$(4, 4, 6)$	$G(48, 19)$	1	0	1	S-NA-NP

Table 7.5.7 Rotation and Tiling Groups, Genus 2-13 - part 7

σ	$\|G\|$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
9	48	$(4,4,6)$	$G(48,30)$	2	0	2	S-NA-NP
9	48	$(4,4,6)$	$G(48,48)$	1	0	1	S-NA-NP
9	57	$(3,3,19)$	$G(57,1)$	0	1	1	S-NA-NP
9	60	$(3,5,5)$	$G(60,5)$	2	0	2	NS
9	64	$(2,8,8)$	$G(64,4)$	1	0	1	p-NA
9	64	$(2,8,8)$	$G(64,6)$	1	0	1	p-NA
9	64	$(2,8,8)$	$G(64,10)$	1	0	1	p-NA
9	64	$(2,8,8)$	$G(64,12)$	1	0	1	p-NA
9	64	$(2,8,8)$	$G(64,36)$	1	0	1	p-NA
9	64	$(4,4,4)$	$G(64,23)$	1	0	1	p-NA
9	64	$(4,4,4)$	$G(64,34)$	1	0	1	p-NA
9	64	$(4,4,4)$	$G(64,35)$	2	0	2	p-NA
9	72	$(2,4,36)$	$G(72,5)$	1	0	1	S-NA-NP
9	80	$(2,4,20)$	$G(80,14)$	1	0	1	S-NA-NP
9	96	$(2,4,12)$	$G(96,13)$	1	0	1	S-NA-NP
9	96	$(2,4,12)$	$G(96,186)$	1	0	1	S-NA-NP
9	96	$(2,4,12)$	$G(96,187)$	1	0	1	S-NA-NP
9	96	$(2,6,6)$	$G(96,70)$	1	0	1	S-NA-NP
9	96	$(3,3,6)$	$G(96,3)$	1	0	1	S-NA-NP
9	96	$(3,4,4)$	$G(96,67)$	1	0	1	S-NA-NP
9	96	$(3,4,4)$	$G(96,227)$	1	0	1	S-NA-NP
9	120	$(2,5,6)$	$G(120,34)$	1	0	1	NS
9	120	$(2,5,6)$	$G(120,35)$	1	0	1	NS
9	128	$(2,4,8)$	$G(128,75)$	1	0	1	p-NA
9	128	$(2,4,8)$	$G(128,134)$	1	0	1	p-NA
9	128	$(2,4,8)$	$G(128,136)$	1	0	1	p-NA
9	128	$(2,4,8)$	$G(128,138)$	1	0	1	p-NA
9	160	$(2,5,5)$	$G(160,199)$	1	0	1	S-NA-NP
9	192	$(2,3,12)$	$G(192,194)$	1	0	1	S-NA-NP
9	192	$(2,4,6)$	$G(192,955)$	1	0	1	S-NA-NP
9	192	$(2,4,6)$	$G(192,990)$	1	0	1	S-NA-NP
9	320	$(2,4,5)$	$G(320,1582)$	1	0	1	S-NA-NP
10	21	$(21,21,21)$	Z_{21}	2	0	2	C
10	22	$(11,22,22)$	Z_{22}	5	0	5	C

Table 7.5.8 Rotation and Tiling Groups, Genus 2-13 - part 8

σ	$\|G\|$	(l,m,n)	Group	#Kal	#non-Kal	total	Type
10	24	$(12,12,12)$	$G(24,11)$	1	0	1	S-NA-NP
10	24	$(6,24,24)$	Z_{24}	1	0	1	C
10	24	$(8,12,24)$	Z_{24}	2	0	2	C
10	25	$(5,25,25)$	Z_{25}	2	0	2	C
10	27	$(9,9,9)$	$Z_3 \times Z_9$	1	0	1	A2
10	27	$(9,9,9)$	$G(27,4)$	0	1	1	p-NA
10	28	$(4,14,28)$	Z_{28}	1	0	1	C
10	30	$(3,30,30)$	Z_{30}	1	0	1	C
10	30	$(5,6,30)$	Z_{30}	1	0	1	C
10	30	$(6,6,15)$	$G(30,2)$	1	0	1	S-NA-NP
10	33	$(3,11,33)$	Z_{33}	1	0	1	C
10	36	$(3,12,12)$	$Z_3 \times Z_{12}$	1	0	1	A2
10	36	$(3,12,12)$	$G(36,6)$	2	0	2	S-NA-NP
10	36	$(4,6,12)$	$G(36,6)$	1	0	1	S-NA-NP
10	36	$(6,6,6)$	$Z_6 \times Z_6$	1	0	1	A2
10	36	$(6,6,6)$	$G(36,12)$	2	0	2	S-NA-NP
10	40	$(2,40,40)$	Z_{40}	1	0	1	C
10	40	$(4,4,20)$	$G(40,4)$	1	0	1	S-NA-NP
10	42	$(2,21,42)$	Z_{42}	1	0	1	C
10	42	$(3,6,14)$	$G(42,2)$	0	1	1	S-NA-NP
10	44	$(2,22,22)$	$Z_2 \times Z_{22}$	1	0	1	A2
10	44	$(4,4,11)$	$G(44,1)$	1	0	1	S-NA-NP
10	48	$(2,12,24)$	$G(48,26)$	1	0	1	S-NA-NP
10	54	$(2,9,18)$	$G(54,4)$	1	0	1	S-NA-NP
10	54	$(3,6,6)$	$G(54,5)$	2	0	2	S-NA-NP
10	54	$(3,6,6)$	$G(54,10)$	1	0	1	S-NA-NP
10	54	$(3,6,6)$	$G(54,12)$	1	0	1	S-NA-NP
10	60	$(2,6,30)$	$G(60,10)$	1	0	1	S-NA-NP
10	63	$(3,3,21)$	$G(63,3)$	0	1	1	S-NA-NP
10	72	$(2,6,12)$	$G(72,23)$	1	0	1	S-NA-NP
10	72	$(2,6,12)$	$G(72,28)$	1	0	1	S-NA-NP
10	72	$(2,6,12)$	$G(72,30)$	1	0	1	S-NA-NP
10	72	$(2,8,8)$	$G(72,39)$	0	1	1	S-NA-NP
10	72	$(3,3,12)$	$G(72,25)$	1	0	1	S-NA-NP
10	72	$(3,4,6)$	$G(72,42)$	1	0	1	S-NA-NP

Table 7.5.9 Rotation and Tiling Groups, Genus 2-13 - part 9

σ	$\|G\|$	(l, m, n)	Group	#Kal	#non-Kal	total	Type
10	72	$(4, 4, 4)$	$G(72, 41)$	0	2	2	S-NA-NP
10	80	$(2, 4, 40)$	$G(80, 6)$	1	0	1	S-NA-NP
10	81	$(3, 3, 9)$	$G(81, 7)$	1	0	1	p-NA
10	81	$(3, 3, 9)$	$G(81, 9)$	1	0	1	p-NA
10	88	$(2, 4, 22)$	$G(88, 7)$	1	0	1	S-NA-NP
10	108	$(2, 4, 12)$	$G(108, 15)$	1	0	1	S-NA-NP
10	108	$(2, 6, 6)$	$G(108, 17)$	1	0	1	S-NA-NP
10	108	$(2, 6, 6)$	$G(108, 25)$	1	0	1	S-NA-NP
10	108	$(2, 6, 6)$	$G(108, 38)$	1	0	1	S-NA-NP
10	108	$(3, 3, 6)$	$G(108, 22)$	1	0	1	S-NA-NP
10	108	$(3, 4, 4)$	$G(108, 15)$	1	0	1	S-NA-NP
10	108	$(3, 4, 4)$	$G(108, 37)$	1	0	1	S-NA-NP
10	144	$(2, 3, 24)$	$G(144, 122)$	1	0	1	S-NA-NP
10	144	$(2, 4, 8)$	$G(144, 182)$	0	1	1	S-NA-NP
10	162	$(2, 3, 18)$	$G(162, 14)$	1	0	1	S-NA-NP
10	168	$(2, 4, 7)$	$G(168, 42)$	1	0	1	NS
10	180	$(2, 3, 15)$	$G(180, 19)$	1	0	1	NS
10	216	$(2, 3, 12)$	$G(216, 92)$	1	0	1	S-NA-NP
10	216	$(2, 4, 6)$	$G(216, 87)$	1	0	1	S-NA-NP
10	216	$(2, 4, 6)$	$G(216, 158)$	1	0	1	S-NA-NP
10	216	$(3, 3, 4)$	$G(216, 153)$	1	1	2	S-NA-NP
10	324	$(2, 3, 9)$	$G(324, 160)$	1	0	1	S-NA-NP
10	360	$(2, 4, 5)$	$G(360, 118)$	1	0	1	NS
10	432	$(2, 3, 8)$	$G(432, 734)$	0	1	1	S-NA-NP
11	23	$(23, 23, 23)$	Z_{23}	4	0	4	C
11	24	$(12, 24, 24)$	Z_{24}	2	0	2	C
11	30	$(6, 10, 15)$	Z_{30}	1	0	1	C
11	32	$(4, 16, 16)$	$Z_2 \times Z_{16}$	1	0	1	A2
11	32	$(4, 16, 16)$	$G(32, 17)$	1	0	1	p-NA
11	32	$(8, 8, 8)$	$G(32, 15)$	2	0	2	p-NA
11	33	$(3, 33, 33)$	Z_{33}	1	0	1	C
11	40	$(4, 8, 8)$	$G(40, 3)$	0	2	2	S-NA-NP
11	44	$(2, 44, 44)$	Z_{44}	1	0	1	C
11	44	$(4, 4, 22)$	$G(44, 1)$	1	0	1	S-NA-NP

Table 7.5.10 Rotation and Tiling Groups, Genus 2-13 - part 10

σ	$\|G\|$	(l, m, n)	Group	#Kal	#non-Kal	total	Type
11	46	$(2, 23, 46)$	Z_{46}	1	0	1	C
11	48	$(2, 24, 24)$	$Z_2 \times Z_{24}$	1	0	1	A2
11	48	$(2, 24, 24)$	$G(48, 24)$	1	0	1	S-NA-NP
11	48	$(3, 8, 8)$	$G(48, 28)$	1	0	1	S-NA-NP
11	48	$(3, 8, 8)$	$G(48, 29)$	1	0	1	S-NA-NP
11	48	$(4, 4, 12)$	$G(48, 11)$	1	0	1	S-NA-NP
11	48	$(4, 4, 12)$	$G(48, 12)$	1	0	1	S-NA-NP
11	48	$(4, 4, 12)$	$G(48, 13)$	1	0	1	S-NA-NP
11	48	$(4, 6, 6)$	$G(48, 32)$	3	0	3	S-NA-NP
11	60	$(2, 12, 12)$	$G(60, 6)$	0	1	1	S-NA-NP
11	60	$(4, 4, 6)$	$G(60, 7)$	0	1	1	S-NA-NP
11	64	$(2, 8, 16)$	$G(64, 40)$	1	0	1	p-NA
11	64	$(2, 8, 16)$	$G(64, 42)$	1	0	1	p-NA
11	66	$(2, 6, 33)$	$G(66, 2)$	1	0	1	S-NA-NP
11	80	$(2, 8, 8)$	$G(80, 28)$	0	1	1	S-NA-NP
11	80	$(2, 8, 8)$	$G(80, 29)$	0	1	1	S-NA-NP
11	80	$(4, 4, 4)$	$G(80, 30)$	0	1	1	S-NA-NP
11	80	$(4, 4, 4)$	$G(80, 31)$	0	1	1	S-NA-NP
11	88	$(2, 4, 44)$	$G(88, 4)$	1	0	1	S-NA-NP
11	96	$(2, 4, 24)$	$G(96, 28)$	1	0	1	S-NA-NP
11	96	$(2, 4, 24)$	$G(96, 32)$	1	0	1	S-NA-NP
11	96	$(2, 6, 8)$	$G(96, 189)$	1	0	1	S-NA-NP
11	96	$(2, 6, 8)$	$G(96, 190)$	1	0	1	S-NA-NP
11	120	$(2, 4, 12)$	$G(120, 36)$	0	1	1	S-NA-NP
11	120	$(2, 6, 6)$	$G(120, 34)$	1	0	1	NS
11	120	$(3, 4, 4)$	$G(120, 34)$	1	0	1	NS
11	160	$(2, 4, 8)$	$G(160, 82)$	0	1	1	S-NA-NP
11	160	$(2, 4, 8)$	$G(160, 85)$	0	1	1	S-NA-NP
11	240	$(2, 4, 6)$	$G(240, 189)$	1	0	1	NS
12	25	$(25, 25, 25)$	Z_{25}	3	0	3	C
12	26	$(13, 26, 26)$	Z_{26}	6	0	6	C
12	27	$(9, 27, 27)$	Z_{27}	3	0	3	C
12	28	$(14, 14, 14)$	$Z_2 \times Z_{14}$	2	0	2	A2
12	28	$(7, 28, 28)$	Z_{28}	3	0	3	C

Table 7.5.11 Rotation and Tiling Groups, Genus 2-13 - part 11

σ	$\lvert G\rvert$	(l, m, n)	Group	#Kal	#non-Kal	total	Type
12	30	$(10, 10, 15)$	$G(30, 1)$	2	0	2	S-NA-NP
12	30	$(5, 30, 30)$	Z_{30}	2	0	2	C
12	30	$(6, 15, 30)$	Z_{30}	1	0	1	C
12	32	$(4, 32, 32)$	Z_{32}	1	0	1	C
12	35	$(5, 7, 35)$	Z_{35}	1	0	1	C
12	36	$(3, 36, 36)$	Z_{36}	1	0	1	C
12	36	$(4, 9, 36)$	Z_{36}	1	0	1	C
12	36	$(6, 9, 9)$	$G(36, 3)$	1	0	1	S-NA-NP
12	39	$(3, 13, 39)$	Z_{39}	1	0	1	C
12	40	$(4, 10, 10)$	$G(40, 10)$	1	0	1	S-NA-NP
12	40	$(5, 8, 8)$	$G(40, 1)$	1	0	1	S-NA-NP
12	40	$(5, 8, 8)$	$G(40, 3)$	0	1	1	S-NA-NP
12	42	$(3, 14, 14)$	$G(42, 3)$	1	0	1	S-NA-NP
12	42	$(6, 6, 7)$	$G(42, 1)$	0	1	1	S-NA-NP
12	42	$(6, 6, 7)$	$G(42, 2)$	0	1	1	S-NA-NP
12	42	$(6, 6, 7)$	$G(42, 4)$	1	0	1	S-NA-NP
12	48	$(2, 48, 48)$	Z_{48}	1	0	1	C
12	48	$(4, 4, 24)$	$G(48, 8)$	1	0	1	S-NA-NP
12	48	$(4, 6, 8)$	$G(48, 16)$	1	0	1	S-NA-NP
12	48	$(4, 6, 8)$	$G(48, 28)$	1	0	1	S-NA-NP
12	50	$(2, 25, 50)$	Z_{50}	1	0	1	C
12	52	$(2, 26, 26)$	$Z_2 \times Z_{26}$	1	0	1	A2
12	52	$(4, 4, 13)$	$G(52, 1)$	1	0	1	S-NA-NP
12	52	$(4, 4, 13)$	$G(52, 3)$	0	1	1	S-NA-NP
12	55	$(5, 5, 5)$	$G(55, 1)$	0	2	2	S-NA-NP
12	56	$(2, 14, 28)$	$G(56, 9)$	1	0	1	S-NA-NP
12	60	$(2, 10, 30)$	$G(60, 11)$	1	0	1	S-NA-NP
12	60	$(2, 15, 15)$	$G(60, 9)$	1	0	1	S-NA-NP
12	80	$(2, 8, 10)$	$G(80, 15)$	1	0	1	S-NA-NP
12	84	$(2, 6, 14)$	$G(84, 8)$	1	0	1	S-NA-NP
12	84	$(3, 3, 14)$	$G(84, 11)$	0	1	1	S-NA-NP
12	96	$(2, 4, 48)$	$G(96, 7)$	1	0	1	S-NA-NP
12	104	$(2, 4, 26)$	$G(104, 8)$	1	0	1	S-NA-NP
12	110	$(2, 5, 10)$	$G(110, 1)$	0	2	2	S-NA-NP
12	120	$(2, 4, 15)$	$G(120, 38)$	1	0	1	S-NA-NP

Table 7.5.12 Rotation and Tiling Groups, Genus 2-13 - part 12

σ	$\|G\|$	(l, m, n)	Group	#Kal	#non-Kal	total	Type
13	27	$(27, 27, 27)$	Z_{27}	2	0	2	C
13	28	$(14, 28, 28)$	Z_{28}	3	0	3	C
13	30	$(10, 15, 30)$	Z_{30}	3	0	3	C
13	32	$(8, 16, 16)$	$Z_2 \times Z_{16}$	2	0	2	A2
13	32	$(8, 16, 16)$	$G(32, 17)$	2	0	2	p-NA
13	36	$(4, 18, 36)$	Z_{36}	1	0	1	C
13	36	$(6, 12, 12)$	$Z_3 \times Z_{12}$	1	0	1	A2
13	36	$(6, 12, 12)$	$G(36, 6)$	2	0	2	S-NA-NP
13	36	$(9, 9, 9)$	$G(36, 3)$	1	0	1	S-NA-NP
13	39	$(3, 39, 39)$	Z_{39}	1	0	1	C
13	40	$(4, 10, 20)$	$Z_2 \times Z_{20}$	1	0	1	A2
13	42	$(3, 14, 42)$	Z_{42}	1	0	1	C
13	45	$(3, 15, 15)$	$Z_3 \times Z_{15}$	1	0	1	A2
13	48	$(3, 12, 12)$	$G(48, 31)$	1	0	1	S-NA-NP
13	48	$(3, 12, 12)$	$G(48, 33)$	1	0	1	S-NA-NP
13	48	$(4, 6, 12)$	$G(48, 21)$	1	0	1	S-NA-NP
13	48	$(4, 6, 12)$	$G(48, 31)$	1	0	1	S-NA-NP
13	48	$(6, 6, 6)$	$G(48, 32)$	1	0	1	S-NA-NP
13	52	$(2, 52, 52)$	Z_{52}	1	0	1	C
13	52	$(4, 4, 26)$	$G(52, 1)$	1	0	1	S-NA-NP
13	54	$(2, 27, 54)$	Z_{54}	1	0	1	C
13	56	$(2, 28, 28)$	$Z_2 \times Z_{28}$	1	0	1	A2
13	56	$(4, 4, 14)$	$G(56, 6)$	1	0	1	S-NA-NP
13	60	$(5, 5, 5)$	$G(60, 5)$	1	0	1	NS
13	64	$(2, 16, 16)$	$G(64, 29)$	1	0	1	p-NA
13	64	$(2, 16, 16)$	$G(64, 30)$	1	0	1	p-NA
13	64	$(2, 16, 16)$	$G(64, 31)$	1	0	1	p-NA
13	64	$(4, 4, 8)$	$G(64, 8)$	1	0	1	p-NA
13	64	$(4, 4, 8)$	$G(64, 9)$	2	0	2	p-NA
13	64	$(4, 4, 8)$	$G(64, 18)$	1	0	1	p-NA
13	64	$(4, 4, 8)$	$G(64, 20)$	1	0	1	p-NA
13	64	$(4, 4, 8)$	$G(64, 21)$	1	0	1	p-NA
13	64	$(4, 4, 8)$	$G(64, 32)$	1	0	1	p-NA
13	64	$(4, 4, 8)$	$G(64, 33)$	2	0	2	p-NA
13	72	$(2, 12, 12)$	$G(72, 21)$	1	0	1	S-NA-NP

Table 7.5.13 Rotation and Tiling Groups, Genus 2-13 - part 13

σ	$\lvert G\rvert$	(l, m, n)	Group	#Kal	#non-Kal	total	Type
13	72	$(2, 12, 12)$	$G(72, 27)$	1	0	1	S-NA-NP
13	72	$(2, 9, 18)$	$G(72, 16)$	1	0	1	S-NA-NP
13	72	$(3, 4, 12)$	$G(72, 42)$	1	0	1	S-NA-NP
13	72	$(3, 6, 6)$	$G(72, 44)$	1	0	1	S-NA-NP
13	72	$(3, 6, 6)$	$G(72, 47)$	2	0	2	S-NA-NP
13	72	$(4, 4, 6)$	$G(72, 45)$	1	0	1	S-NA-NP
13	78	$(2, 6, 39)$	$G(78, 4)$	1	0	1	S-NA-NP
13	90	$(2, 6, 15)$	$G(90, 7)$	1	0	1	S-NA-NP
13	96	$(3, 4, 6)$	$G(96, 3)$	1	0	1	S-NA-NP
13	96	$(3, 4, 6)$	$G(96, 68)$	1	0	1	S-NA-NP
13	96	$(3, 4, 6)$	$G(96, 70)$	1	0	1	S-NA-NP
13	96	$(3, 4, 6)$	$G(96, 71)$	0	1	1	S-NA-NP
13	104	$(2, 4, 52)$	$G(104, 5)$	1	0	1	S-NA-NP
13	112	$(2, 4, 28)$	$G(112, 13)$	1	0	1	S-NA-NP
13	120	$(2, 5, 10)$	$G(120, 35)$	1	0	1	NS
13	128	$(2, 4, 16)$	$G(128, 71)$	1	0	1	p-NA
13	128	$(2, 4, 16)$	$G(128, 79)$	1	0	1	p-NA
13	144	$(2, 4, 12)$	$G(144, 115)$	1	0	1	S-NA-NP
13	144	$(3, 3, 6)$	$G(144, 184)$	1	0	1	S-NA-NP
13	180	$(3, 3, 5)$	$G(180, 19)$	1	0	1	NS
13	288	$(2, 3, 12)$	$G(288, 1024)$	1	0	1	S-NA-NP
13	360	$(2, 3, 10)$	$G(360, 121)$	1	0	1	NS

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