AUTOMORPHIC SUBSETS OF THE n-DIMENSIONAL CUBE ARE TRANSLATIONS OF CWATSETS

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Definition 1 *An automorphic subset of the n-dimensional cube is a subset, S, of Z_2^n whose stabilizer in $Z_2 \wr S_n$ acts transitively on S.*

Jones, Klin and Lazebnik [JKL98] showed that automorphic subsets are generalizations of cwatsets by proving that a cwatset is an automorphic subset containing the all zero word $\mathbf{0}$.

Definition 2 *Two automorphic subsets, X and Y, are internally isomorphic if $Y \in Orb(X)$, where $Orb(X) = \{X^\sigma | \sigma \in Z_2 \wr S_n\}$.*

This paper will show that an automorphic subset is the translation of some cwatset and therefore that each automorphic subset is internally isomorphic to a cwatset.

First we will show that automorphic subsets are invariant under internal isomorphism. This result is implied in [JKL98] but never stated or proved.

Lemma 1 *The orbit of an automorphic subset under the action of $Z_2 \wr S_n$ contains only automorphic subsets.*

Proof: Let X be an automorphic subset, with $Y \in Orb(X)$. Then $H = Stab(X)$ acts transitively on X. Since X and Y are internally isomorphic, there exists some $\sigma \in Z_2 \wr S_n$ such that $X^\sigma = Y$. Therefore, for any $\beta \in H$ we have,

$$Y^{\sigma^{-1}\beta\sigma} = X^{\beta\sigma} = X^\sigma = Y.$$

So it follows that the subgroup $H' = \sigma H \sigma^{-1}$ of $Z_2 \wr S_n$ is a subgroup of $Stab(Y)$. Therefore, if we can show that H' acts transitively on Y, it will follow that $Stab(Y)$ acts transitively on Y and that Y is an automorphic subset.

Let us index the elements of X and Y such that x_i denotes the i^{th} element of X and that $x_i^\sigma = y_i$. It will suffice to show that for all $i, j \leq |Y|$, there exists a $\beta' \in H'$ such that $y_i^{\beta'} = y_j$. Given such an i and j, there exists a $\beta \in H$ such that $x_i^\beta = x_j$, because H acts transitively on X. Consider $\sigma^{-1}\beta\sigma \in H'$:

$$y_i^{\sigma^{-1}\beta\sigma} = x_i^{\beta\sigma} = x_j^\sigma = y_j.$$

That is, H' acts transitively on Y. □

Theorem 2 *X is an automorphic subset if and only if there exists a binary word, b, and some cwatset, C, such that $X = C + b$.*

Proof: Let C be a cwatset and b be a binary word. Consider the element $(id, b) \in Z_2 \wr S_n$. C is an automorphic subset and $C^{(id,b)} \in Orb(C)$, so the lemma implies that $C^{(id,b)} = C + b$ is an automorphic subset. Therefore any translation of a cwatset is an automorphic subset.

Let X be an automorphic subset. Consider an element, $x \in X$. Then $X = (X + x) + x$. We know that $(X + x) = X^{(id,x)}$, so the lemma implies that $(X + x)$ is an automorphic subset. Thus all that remains to be shown is that $\mathbf{0} \in (X + x)$:

$$x \in X \Rightarrow (x + x) \in (X + x) \Rightarrow \mathbf{0} \in (X + x).$$

Therefore, $(X + x)$ is a cwatset and X is the translation of a cwatset. □

Corollary 3 *Every automorphic subset is internally isomorphic to some cwatset.*

Proof: Given an automorphic subset, X, by the theorem there exists a cwatset, C and a binary word, b, such that $X = C + b$. It follows that,

$$X = C^{(id,b)} \Rightarrow X \in Orb(C).$$

Thus, X and C are internally isomorphic. □

References

[JKL98] Gareth Jones, Mikhail Klin, and Felix Lazebnik. Introduction to the theory of automorphic subsets of the n-dimensional cube. Technical Report 309, University of Southampton, August 1998.