The full automorphism group of a family of generalized Fermat curves

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joint work in progress with

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Generalised Fermat Curves of type (k,n)

(named hyper-Fermat by Broughton and Wootton)

$$F = F(\lambda_3, ..., \lambda_n) = \begin{cases} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_3 x_0^k + x_1^k + x_3^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_n x_0^k + x_1^k + x_n^k & = & 0 \end{cases}$$

- Fermat group: $\mathbb{Z}_k^n = H = \langle \tau_1, \cdots, \tau_n \rangle$

$\tau_j[x_0 : \cdots : x_n] = [x_0 : \cdots : \xi_k x_j : \cdots : x_n]$

(with $\xi_k = exp(2\pi i/k)$ and $\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n = 1$)

- Quotient of signature $(0; n+1; k, \cdots, k)$

$$\begin{array}{ccc} F(\lambda_3, ..., \lambda_n)/\mathbb{Z}_k^n & \simeq & (\mathbb{P}^1, \{0, 1, \infty, \lambda_3, \cdots \lambda_n\}) \\ [x_0 : \cdots : x_n] & \leftrightarrow & -(x_1/x_0)^k \end{array}$$

- Genus: $g_{k,n} = 1 + \dfrac{k^{n-1}((n-1)k - n - 1)}{2}$

- G.F.C. of type $(k, 2) \Leftrightarrow$ Classical F.C.

Aut(F) (assuming H is normal)

- Since $F/H \equiv (\mathbb{P}^1, \{0, 1, \infty, \lambda_3, \cdots, \lambda_n\})$ and we are assuming $H \lhd Aut(F)$:

$$1 \to H \to Aut(F) \to \mathbb{M} \to 1$$

where $\mathbb{M} < M\ddot{o}b(\{0, 1, \infty, \lambda_3, \cdots \lambda_n\}) < S_n$, i.e.

Proposition. $Aut(F)$ extension of $H \simeq \mathbb{Z}_k^n$ by certain group $\mathbb{M} < S_n \cap \{\mathbb{Z}_k, D_k, A_4, S_4, A_5\}$

This is, of course, a well-known situation:

- $H = \mathbb{Z}_2$ (hyperelliptic curves)

Brandt-Stichtenoth, Bujalance-Gamboa-Gromadzki, Cirre,...

- $H = \mathbb{Z}_p$ (p-gonal curves $with\ n > 2p \Rightarrow \mathbb{Z}_p$ unique)

Kontogeorgis, Wootton, ...

- $H = \mathbb{Z}_p$ and $gen(F/H) = \gamma > 0$ ((p, γ)-gonal curves with $gen(F) > 2p\gamma + (p-1)^2 \Rightarrow C_p\ unique$)

Gromadzki, Weaver, Wootton,...

- $H = \mathbb{Z}_p$, but non normal ($\Rightarrow n$ small)

Accola, Kallel, Sjerve, Wootton...

G.F.C's of type (k,n): Uniformization

$$F/H \equiv (\mathbb{P}^1, \{\infty, 0, 1..\lambda_n\}) = \mathbb{H}/\Gamma \Rightarrow \begin{cases} F = \mathbb{H}/[\Gamma,\Gamma] \\ \\ H = \Gamma/\,[\Gamma,\Gamma] \end{cases}$$

$(\Gamma = [0; k..k] = \{\gamma_0, .., \gamma_n : \gamma_0^k = \cdots = \gamma_n^k = \gamma_0\gamma_1..\gamma_n = 1\})$

- The orbifold determines the G.F. Pair (F,H)
(another way to say that $(\lambda_3, \cdots \lambda_n) \Rightarrow F(\lambda_3, \cdots \lambda_n)$)

- Every $\tau \in M\ddot{o}b(\{\infty, 0, 1, \cdots \lambda_n\}$ lifts to an automorphism $\tilde{\tau}$ of $\mathbb{H}/\,[\Gamma,\Gamma] \equiv F(\lambda_3, ..., \lambda_n)$

(because, clearly, $Norm(\Gamma) < Norm([\Gamma,\Gamma])$.

- Explicitly:
$\tau\{\lambda_0, \cdots, \lambda_n\} = \{\lambda_{\tau(0)}, \cdots, \lambda_{\tau(n)}\} \Rightarrow$
$\tilde{\tau}\,[x_0 : \cdots : x_n] = \left[c_0 x_{\tau^{-1}(0)} : \cdots : c_n x_{\tau^{-1}(n)}\right]$). So

- $1 \to H \hookrightarrow Aut(F) \twoheadrightarrow M\ddot{o}b(\{\infty, 0, \cdots, \lambda_n\}) \to 1$

Conclusion: $Aut(F)$ is an extension of $H \simeq \mathbb{Z}_k^n$ by the whole group $\mathbb{M} = M\ddot{o}b(\{\infty, 0, 1, \cdots, \lambda_n\})$. Its elements can be explicitly written down.

But, when is H normal?

- **Conjecture**: The Fermat group H of any G.F.C. is **unique** ($\Rightarrow$ normal)

- **Known to be true** in the following cases:

1) $n = 2$ (Classical F.C.: $F_k = X_0^k + X_1^k + X_2^k$)

$Aut(F_k) = \mathbb{Z}_k^2 \rtimes M\ddot{o}b(\{0, 1, \infty\}) = \mathbb{Z}_k^2 \rtimes S_3$
(S_3 permutes/anti-permutes coordinates/ $\{\infty, 0, 1, \}$)

2) k prime and large compared to n (Hidalgo-Leyton).

3) **Theorem.** H is unique for $n = 3$ and any k,i.e. for all G. F. curves of the form

$$F(\lambda) = \left\{ \begin{array}{rcl} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda x_0^k + x_1^k + x_3^k & = & 0 \end{array} \right.$$

(however our proof requires proving first that H is normal)

Aut of $F(\lambda) = \begin{cases} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda x_0^k + x_1^k + x_3^k & = & 0 \end{cases}$

The automorphisms of $F/H = \mathbb{P}^1, \{0, 1, \infty, \lambda\}$
$\alpha(z) = \lambda/z$ and $\beta(z) = (z - \lambda)/(z - 1)$

lift to

$\widehat{\alpha}[x_1 : x_2 : x_3 : x_4] = [x_2 : \lambda^{1/k} x_1 : x_4 : \lambda^{1/k} x_3]$
$\widehat{\beta}[x_1 : .. : x_4] = [(-1)^{1/k} x_3 : x_4 : (\lambda - 1)^{1/k} x_1 : (1 - \lambda)^{1/k} x_2]$

- $H < Aut(S)$ of signature $(k, k, k, k) \Rightarrow$
$\langle H, \widehat{\alpha}, \widehat{\beta} \rangle < Aut(S)$ with signature $(2, 2, 2, k)$ i.e. $[k, k, k, k] \lhd [2, 2, 2, k]$ is in Singerman's list.
($k \geq 3$ odd $\Rightarrow \langle H, \widehat{\alpha}, \widehat{\beta} \rangle \cong \mathbb{Z}_k^3 \rtimes \mathbb{Z}_2^2$, (Schur-Zassenhaus))

The behaviour of the full group $Aut(F(\lambda))$ imitates the situation in genus 1: It depends on the value of the classical modular function

$$j(\lambda) = \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2 (\lambda - 1)^2}$$

Full group of $F(\lambda)$

- For generic λ, (i.e. $j'(\lambda) \neq 0$)

$Aut(F(\lambda)) = \langle H, \widehat{\alpha}, \widehat{\beta} \rangle; signature = (0; 2, 2, 2, k)$

- For $\lambda = -1, 2, 1/2$, (i.e. $j(\lambda) = 1$)

$Aut(F(\lambda)) = \langle H, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \rangle; signature = (0; 2, 4, 2k)$

where (for $\lambda = -1$)
$\widehat{\gamma}[x_1 : x_2 : x_3 : x_4] = [x_3 : x_4 : 2^{1/k} x_2 : (-2)^{1/k} x_1]$

- For $\lambda = (1 \pm i\sqrt{3})/2$, (i.e. $j(\lambda) = 0$)

$Aut(F(\lambda)) = \langle H, \widehat{\alpha}, \widehat{\beta}, \widehat{\delta} \rangle; signature = (0; 2, 3, 3k)$

where (for $\lambda = (1 + i\sqrt{3})/2$)
$\widehat{\delta}[x_1 : x_2 : x_3 : x_4] = [\lambda^{1/k} x_1 : (-1)^{1/k} x_4 : x_2 : x_3]$

Idea of proof: $[0; 2, 2, 2, k]$ can only be contained in $[0; 2, 4, 2k]$ and $[0; 2, 3, 3k]$. In turn, this implies that H is normal and even unique.

Applications to fields of moduli and definition of GFC's of type $(k,3)$

As in the genus 1 case:

- $F(\lambda) \simeq F(\mu) \Leftrightarrow j(\lambda) = j(\mu)$ where j is the classical modular function

$$j(\lambda) = \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$$

- Moduli of GFC's of type $(k,3) = \frac{\mathbb{C}\backslash\{0,1\}}{\Sigma_3} = \mathbb{C}$

- $Gal(\overline{\mathbb{Q}})$ acts faithfully on G.F.C's of type $(k,3)$

- Field of moduli of $F(\lambda)$ equals $\mathbb{Q}(j(\lambda))$

- Field of definition of $F(\lambda)$ equals $\mathbb{Q}(j(\lambda))$?