NON VANISHING DIVISORS FOR GENERAL CYCLIC COVERS AND THEIR THOMAE FORMULA

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ABSTRACT. Let X be a general cyclic cover of $\mathbb{CP}^1$ ramified at m points, $\lambda_1...\lambda_m$. we define a class of non positive divisors on X of degree $g-1$ supported in the pre images of the branch points on X, such that the the standard theta function doesn't vanish on their image in $J(X)$. These divisors generalize the divisors introduced in [BR] and [Na]. Generalizing the results of [BR],[Na] and [EG] we show that up to a certain determinant of the non standard periods of X, the value of the theta functions at these divisors is a polynomial in the branch point of the curve X. Our treatment is based on a generalization of Accola's results of the 3 cyclic sheeted cover [Ac1] and a straightforward generalization of Nakayashiki's approach explained in [Na] in the non singular case for any singular cyclic cover.

1. INTRODUCTION

Let $\phi : X \mapsto \mathbb{CP}^1$ be a cyclic cover of the sphere of degree N given by the equation:

$$y^N = \prod_{i=}^{m} (x - \lambda_i)^{R_i}$$

such that $(R_i, N) = 1$. Choose a base point z_0, a normalized homology basis $a_1...a_g, b_1...b_g$ and a normalized holomorphic differentials $v_1, ...v_g$ to define the Jacobian of $J(X)$ and a standard map $u : X \mapsto J(X)$. Finally define K_{z_0} to be the Riemann constant and τ is the period matrix associated with the homology basis. Recall that J(X) is completely specified by its period matrix τ. We show the following theorem:

Theorem 1.1. *Let r be a total ramification of $\phi : X \mapsto \mathbb{CP}^1$. Select an integer vector $\beta = (beta_1...\beta_m)$ such that:*

1. $0 \leq \beta_i \leq N - 1$
2. *for* $0 \leq k \leq N - 1$, $\sum_{i=1}^{m} (\beta_i + kR_i) mod N = \frac{r}{2}$

Let K_{z_0} be the Riemann's constant, Then

$$\theta \left[u(\sum_{i=1}^{m} \beta_i P_i) + K_{z_0} - u(\sum_{j=1}^{N} \infty_j) \right] (0, \tau) \neq 0$$

and there exists a complex number C not depending on τ such that:

$$\theta \left[u(\sum_{i=1}^{m} \beta_i P_i) + K_{z_0} - u(\sum_{i=1}^{N} \infty_i) \right] (0, \tau) = C det(A) \prod_{i,j=1, i \neq j}^{m} (\lambda_i - \lambda_j)^{\gamma_{ij} + \beta_{ij}}$$

Date:

where detA is a certain determinant of the $g \times g$ matrix of non normalized holomorphic differentials evaluated at $a_i, 1 \leq i \leq g$ and

$$\gamma_{ij} = \sum_{w=0}^{N-1} \{wR_i/N\} \{wR_j/N\},$$

$$\beta_{ij} = \sum_{k=0}^{N-1} \{(\beta_i + kR_i)/N\} \times \{(\beta_j + kR_j)/N\}$$

and $\{\alpha\}$ is the fractional part of α.

the theorem is a generalization of the work started by [BR],[Na] and [EG]. Bershadsky and Radul generalized the Thomae formula for hyper-elliptic covers for a non singular covers of the sphere i.e. when $R_i = 1$ and the number of branch points m is a multiple of N. [Na] gave a more rigorous proof for the formula suggested by [BR] while [EG] treated a special singular case. In this note we adopt the approach of [Na] to show the theorem stated above. More specifically in the first part of the paper we show that indeed the divisors defined above are non vanishing. In the rest of the paper we modify [Na] to the case when $(R_i, N) = 1$ and $\sum_{i=1}^{m} R_i = 0 mod N$. As far as we know it provides Thomae formula for the most wide class of cyclic covers of the Sphere.

The main idea of [Na] is to produce an integrable differential equation that describes the variation of the logarithm of the theta function with respect to the branch points. This is accomplished by constructing certain analytic quantities of the Riemann surfaces locally (as algebraic expressions supported by the branch points) and comparing them to the global expression as derived in [Fa]. Equating the expansions of the global and the local constructions with the local coordinates around the branch points produces the result. We carry this exact program below. It turns out that the general case isn't much different with the case considered by [Na].

The approach presented here isn't the only one to look for these formulas. In a series of papers Hershel Farkas and his collaborators reproved Thomae's original result and used classical approach of Riemann to write the branch points as of cyclic covers as ratios of theta functions. Consequently they were able to get the β part of the formula. In a book currently written with his student Mazel he generalizes his work avoiding the variational approach used in this note.

2. Non positive divisors on Riemann surface

Let X be a Riemann surface and assume that $D = \sum d_i z_i$ is a divisor (not necessarily positive) on it.

Definition 2.1. $H^0(X, \mathcal{O}(D))$ is the collection of functions $f : X \mapsto \mathbb{CP}^1$ on X such that $div(f) \geq D$. In the [FK] notation this is the space $\mathfrak{R}(D)$.
Let $r(D) = dim H^0(X, \mathcal{O}(D))$.

We seek conditions when $\exists E$ a divisor on X such that $E = \sum e_i x_i$, $e_i \geq 0$ and $D \equiv E$. Assume that D is not a positive divisor (otherwise you can set $E = D$) Then if E is positive and equivalent to D there exists a non constant function f such that $div(f) = E/D$. Therefore $f \in H^0(X, \mathcal{O}(-D))$. (That is f is a function such that $div(f) \geq D$.) Conclude that $r(-D) > 0$. We showed the following:

Lemma 2.2. *Let D be a non positive divisor. Then if there is E a positive divisor such that $E \equiv D$ then $r(-D) > 0$.*

Note that because of Jacobi's inversion theorem if D is a divisor such that $r(-D) > 0$ there is always a positive divisor E of degree $g(X) - 1$ and $D \equiv E$.

Now assume that $degE = g(X) - 1$. Apply Riemann Roch and conclude that: $r(-D) = i(-D)$. Choose a base point z_0 on X and let $u : X \mapsto Jac(X)$ the usual mapping from X into its Jacobian. Let K_{z_0} be the Riemann constant. Then Using Riemann vanishing theorem for theta functions we have the following non vanishing criteria for theta functions:

Lemma 2.3. *Let D, $degD = g - 1$ be a non positive divisor such that $r(-D) = 0$ then $\theta\left(u(D) + K_{z_0}\right) \neq 0$.*

3. Cyclic covers

Let $\phi : X \mapsto \mathbb{CP}^1$ be a cyclic cover of the sphere of order N prime number ramified above m points $\lambda_1 ... \lambda_m$. Then $g(X) = \frac{(n-1)(m-2)}{2}$. The equation of X is:

$$y^N = \prod_{i=}^{m} (x - \lambda_i)^{R_i}.$$

and $R_i \in \{1, 2...N-1\}$. The automorphism of order N is: $T(y, x) = (\omega y, x), \omega^N = 1$. Assume that $\sum_{i=1}^{m} R_i = 0(modN)$. For each $j \in \{1...N-1\}$ define t_j to be the number of R_i such that $R_i = j$. Then $\sum_{i=1}^{N-1} t_j = m$. Associate to X a vector $\alpha \in \mathbb{Z}^m$, such that: $\alpha = (1...1, 2...2, 3...3..., N-1...N-1)$. The index j appears t_j times. Let $P_i \in X$ be the ramification point above λ_i. Since $\sum_{i=1}^{m} R_i = 0 mod(N)$ ∞ has precisely N pre images in X. Let these images be: $(\infty_1, ...\infty_N) = \phi^{-1}(\infty)$.

Definition 3.1. For ξ is a divisor of degree 1 on the sphere which is not a ramification point define $a\xi = \sum_{i=1}^{N} \phi^{-1}\xi$. Extend the map to a divisor of any degree on the sphere

On X select a normalized homology basis a_i, b_j and the set of the normalized canonical differentials ω_i. Choosing a base point $z_0 \in X$ define the mapping: $u : X \mapsto J(X)$.

Definition 3.2. For the base point z_0 define the divisor: ${g_N}^1 = \sum_{i=0}^{N-1} T^i(z_0)$.

Definition 3.3. Let $G_0 \in Jac(X)$ be a point such that $NG_0 = u\left({g_N}^1\right)$.

Since $NP_i - {g_N}^1 = 0$ in the Jacobian conclude that: $u(P_k) = C_k + G_0$ and $NC_k = 0$. Let Δ be the canonical class and let K_{z_0} be the Riemann constant i.e. $-2K_{z_0} = u(\Delta)$. Then $u(\Delta) = (N-1)\sum_{i=1}^{m} u(P_i) - 2u\left({g_N}^1\right)$. Using the definition of C_k rewrite the last expression as:$(N-1)\sum_{i=1}^{m} C_i + 2(g(X)-1)G_0$. Therefore Riemann's constant K_{z_0} equals to:

$$K_{z_0} = -\frac{N-1}{2}\sum_{i=1}^{m} C_i - (g(X)-1)G_0 + E_2$$

and E_2 is a point of order 2. i.e. $2E_2 = 0$. Let $E_1 = E_2 + \frac{N-1}{2}\sum_{i=1}^{m} C_i$ then $E_1 = -K_{z_0} - (g(X)-1)G_0$. if N is an odd number conclude that $2NE_1 = 0$ in J(X). We like to formulate the main theorem which is the adaptation of [Ac2] p.26. This describes the vanishing order of theta functions at certain points of the Jacobian. Let r be a total ramification of f Note that: $r = m(N-1)$.

Theorem 3.4. *Let $\beta = (\beta_1 ... \beta_m) \in (0, ..., N-1)$ such that $\sum \beta_j - \frac{r}{2} = 0 mod N$ (r is always even.) Define a sequence of N numbers $\tau_0 ... \tau_{N-1}$ satisfying the equations:*

$$\sum_{i=1}^{m} \overline{\beta_i + k\alpha_i} = \frac{r}{2} - N\tau_k, 0 \leq k \leq N-1$$

Where $\bar{\jmath}$ is the smallest positive residue of $jmodN$. Then the order of the theta function vanishing on the point $\sum_{i=1}^{m} \beta_i C_i - E_1$ in $Jac(X)$ is $\sum_{i=0}^{N-1} Max(0, \tau_i)$.

Remark 3.5. Since $\overline{\beta} = \beta$, notice:

$$\sum_{i=1}^{m} \beta_i = \frac{r}{2} - N\tau_0$$

Proof 2.4:
The theta function vanishes on the point $\sum_{i=1}^{m} \beta_i C_i - E_1$ if and only if there is a positive divisor of degree $g-1$, ψ such that: $\sum_{i=1}^{m} \beta_i C_i - E_1 = u(\psi) + K_{z_0}$. Use the definition of E_1 and C_i and the formula: $g(X) - 1 = r/2 - N$ to write the last equality as:

$$u(\psi) = \sum_{i=1}^{m} \beta_i C_i - E_1 - K_{z_0} = \sum_{j=1}^{m} \beta_j u(P_j) + (\tau_0 - 1) {g_N}^1$$

Where $\sum_{i=1}^{m} \beta_i = \frac{r}{2} - \tau_0 N$. by 3.5. Let:

$$D_1 = \sum_{i=1}^{m} \beta_i P_i + (\tau_0 - 1) {g_N}^1,$$

D_1 is a divisor (*not necessarily positive*). Its degree given by the next proposition:

Proposition 3.6. *D_1 has degree $g(X) - 1$.*

Proof 2.6:
$deg(D_1) = \sum_{i=1}^{m} \beta_i + N \times (\tau_0 - 1)$ By definition of τ_0:

$$\deg(D_1) = r/2 - N \times \tau_0 + N \times \tau_0 - N$$

Recall that: $2g(X) - 2 = r - 2N$, hence $g - 1 = r/2 - N$ and the proposition follows.■

since D_1 is invariant under T, $\forall f \in H^0(X, \mathcal{O}(-D_1))$, $Tf \in H^0(X, \mathcal{O}(-D_1))$. T is cyclic and unitary therefore $H^0(X, \mathcal{O}(-D_1))$ has a decomposition: $H^0(X, \mathcal{O}(-D_1)) = \bigoplus L_\chi$ and L_χ is the vector space of T eigenvectors with a character: $\chi : Z_n \mapsto \mathbb{C}$. If $N_\chi = dim L_\chi$ then Riemann Theorem applies that the order of vanishing of the theta function is: $N = \sum N_\chi$. We attempt to find N_χ. T is cyclic hence its characters are of the form ω^k for some k, and $\omega^N = 1$. Now $Ty = \omega y$, and $Ty^k = \omega^k y^k$. Conclude that if $f \in N_\chi$ then f/y^k for some k is a pull back of a function g on $\mathbb{CP}^1$. But $f/y^k \in H^0\left(X, \mathcal{O}\left(-D_1 - div(y^k)\right)\right)$. Further f/y^k corresponds to the functions that are pullbacks from the functions on the $\mathbb{CP}^1$ in the space: $H^0\left(X, \mathcal{O}\left(-D_1 - div(y^k)\right)\right)$. Let ${V_k}^0$ be the space of $f \in H^0\left(X, \mathcal{O}\left(-D_1 - div(y^k)\right)\right)$ that are pullbacks from functions on the sphere.

Lemma 3.7. *There is a divisor σ_0 with support on $\mathbb{CP}^1$ such that: $H^0\left(\mathbb{CP}^1, \mathcal{O}(-\sigma_0)\right)$ is isomorphic to ${V_k}^0$.*

Proof 2.7:
For a ramification point λ_j let $\gamma_j = \left[\frac{kR_j+\beta_j}{N}\right] \times N$(i.e. γ_j is the maximal number such that $\gamma_j \leq kR_j + \beta_j$ and $\gamma_j = 0 mod N$.). Let Q_{λ_i} be the point on $\mathbb{CP}^1$ that corresponds to λ_i. Define

$$\sigma_0 = \sum_{j=1}^{m} \frac{\gamma_j}{N} Q_{\lambda_j} + (\tau_0 - 1)\phi(z_0) - k\frac{\sum_{j=1}^{m} R_j}{N}\infty.$$

We show that σ_0 is the desired divisor. Let $h : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$ be a function such that $divh \geq -\sigma_0$. assume that $\hat{h}$ is a lift of h to X. Then

$$div\hat{h} \geq -\gamma_j P_j - (\tau_0 - 1)g_1^N + k\frac{\sum_{j=1}^{m} R_j}{N}\sum_{i=1}^{N} \infty_i.$$

but $\gamma_j \leq kR_j + \beta_j$, and from the definition of ${V_k}^0$ conclude, $\hat{h} \in {V_k}^0$. Now assume that $\hat{f}$, a lift of a function f on the sphere and $\hat{f} \in {V_k}^0$. By definition at a point P_j the $Ord_{P_j}(\hat{f}) \geq -\beta_j - kR_j$. $\hat{f}$ is a lift of f hence, $Ord_{P_j}(\hat{f}) = 0 Mod N$ conclude that $Ord_{P_j}(\hat{f}) \geq \gamma_j$ Or $Ord_{Q_{\lambda_j}}(f) \geq \frac{-\gamma_j}{N}$. In other points of its support The order of $div(-D_1 - y^k)$ is divisible by N. Thus: $f \in H^0\left(\mathbb{CP}^1, \mathcal{O}\left(-\sigma_0\right)\right)$. ■

The immediate conclusion from the lemma is that: $dim{V_k}^0 = dim H^0\left(\mathbb{CP}^1, \mathcal{O}\left(-\sigma_0\right)\right)$. Let us compute the degree of σ_0. By definition of σ_0 we have:

$$deg\sigma_0 = \frac{1}{N} \times \left(\sum_{i=1}^{m} \gamma_i + (\tau_0 - 1)N - \sum_{j=1}^{m} kR_i\right)$$

But $\gamma_i = \beta_i + kR_i - \overline{\beta_i + kR_i}$ by the definition of γ_i. Substituting this expression into γ_i rewrite the last expression as:

$$\frac{1}{N} \times \left(\sum_{i=1}^{m} \beta_i + kR_i - \left(\overline{\beta_i + kR_i}\right) + (\tau_0 - 1)N - \sum_{j=1}^{m} kR_i\right)$$

Cancel kR_i and apply the definition of τ_0 to simplify further:

$$\frac{1}{N} \times \left(\frac{r}{2} - \sum_{i=1}^{m} \left(\overline{\beta_i + kR_i}\right)\right) - 1.$$

By definition of τ_i this equals to: $\tau_k - 1$. Apply Riemann Roch on $\mathbb{CP}^1$ to conclude that $dim H^0\left(\mathbb{CP}^1, \mathcal{O}\left(-\sigma_0\right)\right) = Max(\tau_i, 0)$. ■
The discussion in section 1 produces the following corollary:

Corollary 3.8. *Choose β_i as in theorem but $\tau_i = 0$ then under the conditions of last theorem $\theta\left(\beta_i C_i - E_1\right)$ is not vanishing.*

Recall that $-E_1 - K_{z_0} = -(g(X)-1)G_0$, Or $-E_1 = K_{z_0} + (g(X)-1)G_0$. Rewrite the divisor from corollary as:

$$\sum_{i=1}^{m} \beta_i(P_i - G_0) + K_{z_0} + (g(X) - 1)G_0 = \sum_{i=1}^{m} \beta_i P_i - \beta_i G_0 + K_{z_0} + (g(X) - 1)G_0 =$$

$$\sum_{i=1}^{m} \beta_i P_i - \frac{r}{2}G_0 + (\frac{r}{2} - N)G_0 + K_{z_0}$$

and the last expression is readily seen to be equal to : $\sum_{i=1}^{m}\beta_i P_i + K_{z_0} - \sum_{i=1}^{N}\infty_i$.

Corollary 3.9. *Let β_i be selected such that $\tau_i = 0, 0 \le i \le N-1$ then*

$$\sum_{i=1}^{N}\beta_i P_i + K_{z_0} - \sum_{i=1}^{N}\infty_i$$

is a non vanishing divisor for theta functions.

Remark 3.10. Gabino in [GG] obtained similar results but the theorem stated here seems to be stronger and was independently obtained. See also [EF] for an alternative proof where non positive divisors of degree $g-1$ are replaced with the more traditional special divisors of degree g.

4. N=3 example

We work out the general $N = 3$ example following [Ac1]. Let us represent the curve as : $y^3 = \prod_{i=1}^{s}(x-p_i)\prod_{j=1}^{t}(x-q_j)^2$. and $s + 2t = 0 mod 3$. Then G_0 be a point satisfying $3G_0 = u(z_1+z_2+z_3)$ in the Jacobian. The ramification points lying above p_i and q_j will be respectively: $a_k = A_k + G_0$ and $b_k = B_k + G_0$. Consider the sum $\sum_{i=1}^{s}\epsilon_i A_i + \sum_{j=1}^{t}\delta_j B_j - E_1$. In the vector $\alpha = (1...1, 2...2)$ the ones appearing s times and 2 appearing t times. Then we have the following conditions on ϵ_i, δ_j :

- $\sum \epsilon_i + \sum \delta_j = s + t - 3\tau_0$
- $\overline{\sum(\epsilon_i+1)} + \overline{\sum(\delta_j+2)} = s + t - 3\tau_1$
- $\overline{\sum(\epsilon_i+2)} + \sum(\delta_j+1) = s + t - 3\tau_2$

Now rewrite the period as: $E_1 - \sum_{S_1} A - 2\sum_{S_2} A - \sum_{T_1} B - \sum_{T_2} 2B$ where S_1, S_2 are subsets where appearing 1 and 2 in the A part of the sum and T_1 and T_2 appearing in the B part of the sum. Accordingly $|S_i| = s_i$ and similarly $|T_i| = t_i$. Finally S_0, T_0 be subsets of indices such that $\epsilon_i = \delta_i = 0$. Now define $\mu_0 = s_0 - t_2$, $\mu_1 = s_1 - t_1$ and $\mu_2 = s_2 - t_0$. Then we can write the condition on τ_i as follows: $3\tau_0 = \mu_0 - \mu_2, 3\tau_1 = \mu_2 - \mu_1, 3\tau_2 = \mu_1 - \mu_0$. Thus we obtain: $\mu_0 = \mu_1 = \mu_2 = \frac{(t-s)}{3}$ guarantees non vanishing. We showed:

Theorem 4.1. *Let $\{S_0, S_1, S_2\}, \{T_0, T_1, T_2\}$ be a partition of $a_i, 1 \le i \le s$ and $b_j, 1 \le j \le t$ respectively. if $t_2 = s_0 - (t-s)/3, t_1 = s_1 - (t-s)/3, \tau_0 = s_2 - (t-s)/3$. Then θ does not vanish on the following divisor:*
$(\sum_{S_1} A + 2\sum_{S_2} B + \sum_{T_1} B + 2\sum_{T_2} B) - \infty_1 - \infty_2 - \infty_3 + K_{z_0}$.

5. The non singular case

As a second example of applying the result assume $R_i = 1$. Then $m = pN$. Then we can identify the coefficients β_i with a vector v in the integral lattice: Z^m such that the i-th coordinate of v is β_i. Now the vector $\alpha = (1, 1, 1...1, 1))$ and consequently the τ_k are defined as:

$$\sum_{i=1}^{m}\overline{\beta_i + k} = \frac{r}{2} - N\tau_k.$$

since $r = m(N-1)$ we can rewrite the last expression as :

$$\sum_{i=1}^{m}\overline{\beta_i + k} = m\frac{N-1}{2} - N\tau_k$$

Let t_l be the number of times that $\beta_i = l, 0 < l < N-1$. w.l.o.g we can assume that $max\,(t_1...t_{N-1}) = t_1$ then:

$$\sum_{i=1}^{m} \overline{\beta_i + k} = N\frac{N-1}{2}t_1 + 2(t_2 - t_1) + 3(t_3 - t_1) + ...(N-1)(t_{N-1} - t_1)$$

Because $t_i - t_1 \leq 0$ The condition for non vanishing turns out to be: $t_i = t_1 \forall i$ and $t_1 = p$. We obtained the following theorem:

Theorem 5.1. *Assume that the $R_i = 1, \forall i$ Then the divisor $\sum_{i=1}^{N} \beta_i P_i + K_{z_0} - \sum_{i=1}^{N} \infty_i$ is non vanishing if and only if the number of β_i such that $\beta_i = l, 0 \leq l \leq N-1$ is k.*

6. Properties of divisors

Having established **Corollary 3.9** we show various properties of these divisors useful in the sequel:

Proposition 6.1. *Let $D = \sum_{i=1}^{m} \beta_i P_i$ and let $E = \sum_{i=1}^{m} \left(\overline{\beta_i + k\alpha_i}\right) P_i$. Then $D \equiv E$*

Proof 6.1:
We show that there exists a function on X such that its divisor is: D/E. Define:

$$f = \frac{\prod_{i=1}^{m}(x - \lambda_i)^{\beta_i/N + k\alpha_i/N}}{\prod_{i=1}^{m}(x - \lambda_i)^{\overline{\beta_i + k\alpha_i}/N + k\alpha_i/N}}$$

Clearly the divisor of f is precisely D/E. We verify that this is indeed a function in X. Using the definition of y we rewrite f as : Rewrite f as :

$$y^{-k} \times \frac{\prod_{i=1}^{m}(x - \lambda_i)^{\beta_i/N + k\alpha_i/N}}{\prod_{i=1}^{m}(x - \lambda_i)^{\overline{\beta_i + k\alpha_i}/N}}$$

and by definition $\beta_i + k\alpha_i - \overline{\beta_i + k\alpha_i} = h_i N, h_i \in \mathbb{Z}$. Hence $f = y^{-k} \times \prod_{i=1}^{m} (x - \lambda_i)^{h_i}$ ■

Proposition 6.2. *Let $D = u(\sum_{i=1}^{m} \beta_i P_i) + K_{z_0} - u(\sum_{i=1}^{N} \infty_i)$ as in 3.9. Then $-D = u(\sum_{i=1}^{m}(N - 1 - \beta_i)) + K_{z_0} - u(\sum_{i=1}^{N} \infty_i)$*

Proof 6.2:
We can write:

$$-D = -u(\sum_{i=1}^{m} \beta_i P_i) - K_{z_0} + u(\sum_{i=1}^{N} \infty_i) =$$

$$-u(\sum_{i=1}^{m} \beta_i P_i) - 2K_{z_0} + K_{z_0} + 2\sum_{i=1}^{N} \infty_i - \sum_{j=1}^{N} \infty_i$$

Because of the definition of K_{z_0} we obtain that $-2K_{z_0} \equiv \sum_{i=1}^{m}(N-1)P_i - 2\sum_{j=1}^{m} \infty_i$. Hence:

$$-\sum_{i=1}^{m} \beta_i P_i - 2K_{z_0} + K_{z_0} + 2u(\sum_{i=1}^{N} \infty_i) - u(\sum_{i=1}^{N} \infty_i) =$$

$$\sum_{i=1}^{m}(N-1-\beta_i)P_i + K_{z_0} - \sum_{j=1}^{N}\infty_i$$

■

Lemma 6.3. *For $0 \le k \le N-1$ Let*

$$f_k(z) = \prod_{i=1}^{m}(z-\lambda_i)^{\frac{\overline{\beta_i+k\alpha_i}-(\frac{N-1}{2})}{N}}\sqrt{dz}$$

Then $f_k(z)$ is a meromorphic whose divisor is equivalent to: $\sum_{i=1}^{m}\beta_i P_i - \sum_{i=1}^{N}\infty_i$

Proof 5.3:
The order of $z-\lambda_i$ at P_i is N. Hence the order of the of $(z-\lambda_i)^{\frac{\overline{\beta_i+k\alpha_i}-(\frac{N-1}{2})}{N}}\sqrt{dz}$ is exactly $\overline{\beta_i+k\alpha_i}$. at ∞_i the order is: $\sum_{i=1}^{m}\overline{\beta_i+k\alpha_i} - \frac{r}{2} - 1 = -1$ and hence the divisor is:

$$\sum_{j=1}^{m}\overline{\beta_j+k\alpha_j}P_j - \sum_{i=1}^{N}\infty_i$$

which is equivalent to : $\sum_{j=1}^{m}\beta_j P_j - \sum_{i=1}^{N}\infty_i$. ■

7. Algebraic construction of the Szego Kernel

Let us recall the definition of the Szego Kernel.

Definition 7.1. For $e \in \mathbb{C}^g$. if $\theta[e] \neq 0$ define the Szego kernel by the following equation:

$$S(P,Q|e) = \frac{\theta[e](u(P-Q)}{\theta[e](0,\tau)E(P,Q)}, P, Q \in C.$$

$E(P,Q)$ is the prime form. e depends only on its in the Jacobian , $J(X)$. $R(P,Q|e)$ has the following properties that are well known [F],[EG]:

- $S[e](P,Q)$ is a $\left(\frac{1}{2},\frac{1}{2}\right)$ form with a simple pole along the diagonal
- $S[e]\,(P,Q)$ has divisor $[e-K_{z_0}]$ with respect to variable Q
- $S[e]\,(P,Q)$ has a divisor $[-e-K_{z_0}]$ with respect to variable P.

We generalize the approach of Nakayashiki to give the following expression to the Szego kernel:

Theorem 7.2. *Let $P=(x_1,y_1), Q=(x_2,y_2) \in C$. Choose $\widetilde{\beta} = (\beta_1 ... \beta_m) \in \mathbb{Z}^m$ be as in **3.9**. if $e = \sum_{i=1}^{m}u(\beta_i P_i) + K_{z_0} - u(\sum_{i=1}^{N}\infty_i)$ Let*
(1)

$$F_{\widetilde{\beta}}\,(P,Q) = \frac{1}{N}\frac{\sum_{k=0}^{N-1}\prod_{i=1}^{m}(x_1-\lambda_i)^{\frac{\overline{\beta_i+k\alpha_i}-\frac{N-1}{2}}{N}} \times \prod_{i=1}^{m}(x_2-\lambda_i)^{\frac{\overline{-\beta_i-k\alpha_i}+\frac{N-1}{2}}{N}}\sqrt{dx_1}\sqrt{dx_2}}{x_2-x_1}$$

Then

$$S[e](P,Q) = F_{\widetilde{\beta}}\,(P,Q) \tag{2}$$

Proof 6.2:
We verify that $F_{\widetilde{\beta}}\,(P,Q)$ satisfies the properties characterizing $S[e]\,(P,Q)$. the RHS of the equation **(1)**.

$F_{\widetilde{\beta}}(P,Q)$ is regular outside $P=Q$. If $P\neq Q$ we need to check the case when $x_1=x_2$ and $y_1\neq y_2$. This means that $y_2=\omega^j y_1$. and $\omega^N=1$.
Now, $\overline{\beta_i+k\alpha_i}=\beta_i+k\alpha_i-h_{ki}N$. Rewrite $F_{\widetilde{\beta}}(P,Q)$ as:

$$\frac{1}{N}\times\frac{1}{x_2-x_1}\times\sum_{k=0}^{N-1}\left(\frac{y_1}{y_2}\right)^k\times\left(\frac{x_1-\lambda_i}{x_2-\lambda_i}\right)^{h_{ik}}\prod_{i=1}^{m}\left(\frac{x_1-\lambda_i}{x_2-\lambda_i}\right)^{\frac{\beta_i-\frac{N-1}{2}}{N}}\sqrt{dx_1}\sqrt{dx_2}$$

if $y_2=\omega^k y_1$. In the limit when $x_1\to x_2$, $y_2\to\omega^k y_1$. and $\frac{\sum_{k=0}^{N-1}\left(\frac{y_1}{y_2}\right)^k}{x_2-x_1}\to 0$. Therefore $F_{\widetilde{\beta}}(P,Q)$ is regular when $x_1=x_2$ but $y_1\neq y_2$.
Let us calculate the expansion of

$$\frac{1}{N}\sum_{k=0}^{N-1}\prod_{i=1}^{m}(x_1-\lambda_i)^{\frac{\overline{\beta_i+k\alpha_i}-\frac{N-1}{2}}{N}}\times\prod_{i=1}^{m}(x_2-\lambda_i)^{\frac{\overline{-\beta_i-k\alpha_i}+\frac{N-1}{2}}{N}}$$

as a function of x_2 when the expansion is around x_1. Assuming $x_1=x_2$ we get that the leading coefficient is $\frac{1}{N}\sum_{i=0}^{N-1}1=1$ Now for the coefficient in x_2-x_1 we obtain using the derivative product rule that the coefficient is:
(3)

$$\frac{1}{N}\sum_{i=0}^{N-1}\sum_{j=1}^{m}\left(\overline{-\beta_j-k\alpha_j}/N+\frac{(N-1)}{2N}\right)\times\frac{1}{x_2-\lambda_j}=\frac{1}{N}\sum_{j=1}^{m}\sum_{i=0}^{N-1}\left(\overline{-\beta_j-k\alpha_j}/N)+\frac{r}{2}\right)\times\frac{1}{x_2-\lambda_j}$$

But $\sum_{j=1}^{m}\left(\overline{-\beta_j-k\alpha_j}/N\right)+\frac{r}{2}=0$ and hence

$$\frac{1}{N}\sum_{j=1}^{m}\sum_{i=0}^{N-1}\left(\overline{-\beta_j-k\alpha_j}/N)+\frac{r}{2}\right)\times\frac{1}{x_2-\lambda_j}=0$$

as well. Taking the second derivative according to x_1. to calculate the coefficient of (x_1-x_2) we arrive to the following result:

Proposition 7.3. *The expansion of $F_{\widetilde{\beta}}(P,Q)$ around P a non branch point is:*
(4)

$$F_{\widetilde{\beta}}(P,Q)=\frac{\sqrt{dx_1}\sqrt{dx_2}}{x_2-x_1}\left[1+\frac{1}{2N}\sum_{i,j=1}^{i,j=m}\frac{q(\beta_i,\beta_j)}{(x_2-\lambda_i)(x_2-\lambda_j)}\times(x_1-x_2)^2+...\right]$$

where

$$q(\beta_i,\beta_j)=\sum_{k=0}^{N-1}\{(\beta_i+kR_i)/N\}\times\{(\beta_j+kR_j)/N\}$$

Where x_1,x_2 are the local coordinate around P,Q respectively.

To complete the proof of theorem (7.2) note that $L(P,Q)=F_{\overline{\beta}}(P,Q)-S[e](P,Q)$ are a section of a line bundle $L_{[e+K_{z_0}]}\bigotimes L_{[-e+K_{z_0}]}$. Because of the expansion of $F_{\overline{\beta}}(P,Q)$ conclude that $L(P,Q)$ is a holomorphic section of the line bundle. But

$$H^0\left(L_{[}e+K_{z_0}]\bigotimes L_{[}-e+K_{z_0}]\right)=H^0\left(L_{[e+K_{z_0}]}\right)\bigotimes H^0\left(L_{[-e+K_{z_0}]}\right)=0$$

and thus $F_{\overline{\beta}}(P,Q)=S[e](P,Q)$ as required. ■

Remark 7.4. The above argument is exactly the method adopted in [Na] to show this. See [EG] for a slightly different approach.

Based on the the formula given at the beginning of the section [Na] shows the following expansion for $S[e](P,Q)$ in terms of theta functions:

Corollary 7.5. *The expansion of the Szego kernel can be given in terms of theta functions as follows:*

$$S[e](P,Q)=\frac{\sqrt{dx_1}\sqrt{dx_2}}{x_1-x_2}\times\left[1+\sum_{i=1}^{g}\frac{\partial log\theta[e]}{\partial z_i}(0)u_i(x_1)(x_1-x_2)+...\right]$$

$u_i(x)$ is the coefficient of dx_1 in the expansion of the holomorphic $v_i(x)$.

Comparing the expansions conclude the following result:

Corollary 7.6.

$$\frac{\partial\theta[e]}{\partial z_i}(0)=0$$

The following is obtained by multiplying the expansions:

Lemma 7.7.
(5)

$$S[e](P,Q)S[-e](P,Q)=\frac{dx_1dx_2}{(x_1-x_2)}\left[1+\frac{1}{N}\sum_{i,j=1}^{m}\frac{q\left(\beta_i,\beta_j\right)}{(x_2-\lambda_i)(x_2-\lambda_i)}(x_1-x_2)^2+...\right]$$

8. Algebraic construction for the canonical differential

We construct the canonical differential algebraically for cyclic curves.

Definition 8.1. The canonical symmetric differential is a $\omega(x,y)$ is a meromorphic one differential with respect to $x,y\in C$, having a unique pole of second order when z tends to w with a leading expansion coefficient of 1. Further for a canonical homology basis $a_i,b_j,1\leq i,j\leq g$ we have:

$$\int_{a_i}\omega(x,y)=0$$

for fixed y.

First we remind the reader of a possible basis for the holomorphic differentials on C.

Lemma 8.2. *Let $s_l(z)=\prod_{i=1}^{m}(x-\lambda_i)^{\overline{\frac{lR_i}{N}}}.,0\leq l\leq N-1$ Then a basis for holomorphic differentials is given by:*

$$\frac{z^{j-1}dz}{s_l(z)},$$

where $j=1...d(l),d(l)=Max\left(\sum_{i=1}^{m}\overline{\frac{lR_i}{N}}-1,0\right)$.

Proof 8.2:
The order of $z-\lambda_i$ at λ_i is N. Hence the order of $(z-\lambda_i)^{\overline{\frac{lR_i}{N}}}$ is $\overline{lR_i}<N-1$. Hence we have non trivial 0 at λ_i. For $\infty_i,i=1...n$. The order of $y_l(z)$ is $\sum_{i=1}^{N}\overline{\frac{lR_i}{N}}$. Thus if $j<\sum_{i=1}^{N}\overline{\frac{lR_i}{N}}$ we will not have a pole at ∞_i. ■.
Let,

$$P_l^{(l)}(z,w)=\sum_{n=0}^{d(l)+1}A_n^{(l)}(w)(z-w)^n,$$

such that:

(1) $A_0^{(l)}(w) = \prod_{i=1}^{m}(w-\lambda_i)$
(2) $A_1^{(l)} = \sum_{i=1}^{m} \overline{lR_i}/N \times \frac{A_0^{(l)}(w)}{w-\lambda_i}$
(3) $deg_w P_l^{(l)} \leq d(N-l)+1$

Set:

$$\xi_0(x,y) = \frac{dz(x)dz(y)}{(z(x)-z(y))^2} \tag{6}$$

$$\xi_l(x,y) = \frac{P_l^{(l)}(z(x),z(y))\,dz(x)dz(y)}{s_l(x)s_{N-l}(y)\,(z(x)-z(y))^2} \tag{7}$$

$$\xi(x,y) = \frac{1}{N}\sum_{i=0}^{N-1}\xi_l(x,y) \tag{8}$$

Proposition 8.3. (1) *$\xi(x,y)$ is holomorphic outside the diagonal set $\{x=y\}$.*
(2) *For a non branch point $P \in X$ take z to be a local coordinate around P. Then the expansion in $z(x)$ at $z(y)$ is :*

$$\xi(x,y) = \frac{dz(x)dz(y)}{(z(x)-z(y))^2} + O\left((z(x)-z(y))^0\right)$$

Proof 8.3:
To show the proposition we need first need to show that if $z(P)=z(Q)$ but $P \neq Q$ on X, then $\xi(P,Q)$ is still non singular. Assume that $P=(p_1,q_1)$ and $Q=(p_1,\omega^r q_1)$. Let us examine the leading term of the expansion of $\xi_l(x,y)$ around Q. By definition of $\xi_l(x,y)$ it is:

$$\frac{A_0^{(l)}(z(Q))}{\omega^{rl}q_1^l/\prod_{i=1}^{m}(z-\lambda_i)^{[lR_i/N]}q_1^{N-l}\times(z(P)-z(Q))^2} =$$

$$= \omega^{-rl}\times(z(P)-z(Q))^{-2},$$

(by definition of $A_0^{(l)}$) Then if $r=0$ (i.e. $P \to Q$) the leading coefficient of $\xi(x,y)$ is: $\frac{1}{(z(P)-z(Q))^2}$. if $r>0$, summing we obtain that the coefficient is 0. Next we compute the coefficient of $\frac{1}{z(P)-z(Q)}$in the expansion of $\xi_l(x,y), l>0$. Apply the product rule for derivatives to obtain that the coefficient of $\frac{1}{z(P)-z(Q)}$ is expanding around Q is:

$$\frac{\sum_{i=1}^{m}\{lR_i/N\}A_0^{(l)}(z(Q))}{z(Q)-\lambda_i}s_l(Q) - \sum_{i=1}^{m}\{lR_i/N\}\frac{s_l(z(Q))}{z(Q)-\lambda_i}A_0^{(l)}(z(Q)) = 0$$

therefore the coefficient of $\frac{1}{z(P)-z(Q)}$ is 0 and the proposition is proved.
■

As an immediate corollary of the proposition we have that:

Corollary 8.4.

$$\omega(x,y) - \xi(x,y)$$

is holomorphic on $X \times X$.

Thus by the corollary there exist polynomials $P_k^{(l)}$ such that:

$$\omega(x,y)-\xi(x,y)=\sum_{l=1}^{N-1}\sum_{k=1,k\neq l}^{N-1}\frac{P_k^{(l)}\left(z(x),z(y)\right)dz(x)dz(y)}{s_k(z(x))s_{N-l}(z(y))}$$

Where by modifying the definition of $P_l^{(l)}$ we can exclude the terms $k=l$. as before we can write

$$P_k^{(l)}(z,w)=\sum_{j=0}^{d(k)}A_{kj}^{(l)}(w)(z-w)^j.$$

Note that $deg_w P_k^{(l)}(z,w)\leq d(N-l)$. Our aim is to show the following proposition:

Proposition 8.5.

$$\sum_{l=1}^{N-1}A_{l2}^{l)}\left(\lambda_i\right)=-\prod_{j=1,i\neq j}^{m}(\lambda_i-\lambda_j)\frac{\partial}{\partial\lambda_i}logdetC \tag{9}$$

and C is a $g(X)\times g(X)$ period matrix of non normalized form :

$$\left(\int_{a_i}z^{j-1}dz/s_l(z)\right).$$

Proof 7.5:
Let us take a local $t=(z-\lambda_i)^{\frac{1}{N}}$ coordinate around Q_i. Then we have that the condition that $\int_{a_j}\omega(x,y)=0$ is equivalent to the coefficient of the expansion around Q_i in $dt,tdt,...t^{N-2}dt$ is vanishing. A short calculation shows that this is equivalent to

$$\omega^{(l)}\left(x\right)=\frac{1}{N}\frac{P_l^{(l)}(z(x),\lambda_i)dz(x)}{s_l(x)(z(x)-\lambda_i)^2}+\sum_{k=1,k\neq l}^{N-1}\frac{P_k^{(l)}\left(z(x),\lambda_i\right)dz(x)}{s_k(x)} \tag{10}$$

vanishing when we integrate around a_j. Let us write this explicitly: Note,

$$\frac{\partial}{\partial\lambda_i}\frac{dz}{s_l}=\left\{lR_i/N\right\}\frac{dz}{s_l\left(z-\lambda_i\right)}$$

and

$$P_l^{(l)}(z,\lambda_i)=\left\{lR_i/N\right\}\prod_{j=1}^{m}\left(\lambda_i-\lambda_j\right)\left(z-\lambda_i\right)+\sum_{j=0}^{d(l)-2}A_{l,j+2}^{(l)}(z-\lambda_i)^{j+2}.$$

hence:

$$\frac{\prod_{j=1}^{m}\left(\lambda_i-\lambda_j\right)}{N}\frac{\partial}{\partial\lambda_i}\int_{a_h}\frac{dz}{s_l}+\frac{1}{N}\sum_{j=0}^{d(l)-1}A_{l,j+2}^{(l)}(\lambda_i)\int_{a_h}(z-\lambda_i)^j dz/s_l \tag{11}$$

$$+\sum_{k=1,k\neq l}\sum_{j=0}^{d(k)-1}A_{k,j}^{(l)}(\lambda_i)\int_{a_h}(z-\lambda_i)^j dz/s_k=0$$

Following [BR],[Na](see also [EG] for a slightly different approach.) regard for a fixed l the equations above as $g(X)$ equations in $g(X)$ variables, $A_{k,r}^{(l)}$. The matrix of these equations is the $g(X)\times g(X)$ matrix B,

$$B=\int_{a_h}\left(z-\lambda_i\right)^{j-1}dz/s_l(z),l=1...N-1,j=1...d(l).$$

For each l define matrices B_l obtain from B by replacing the column $\int_{a_h} \frac{dz}{s_l(z)}, 1 \leq g(X)$ with the column: $\frac{\partial}{\partial \lambda_i} \int_{a_h} \frac{dz}{s_l(z)}$. Then by Cramer's rule:

$$A_{l2}^{(l)} = \frac{detB_l}{detB}.$$

Expand $(z-\lambda_i)^k, k \geq 0$ using the binomial formula and perform elementary operations on columns to obtain that $detB = detC$. We now prove a similar proposition for the matrix $detB_l$.

Proposition 8.6.

$$detB_l = \sum_{i=0}^{d(l)} detC_i$$

where C_i is the matrix C where the $i-th$ column is replaced with $\frac{\partial}{\partial \lambda_i} \frac{z^i dz}{s_l(z)}$.

Proof 8.6 Assume inductively that the proposition is true for any minor of B_l that contains the first v columns. The polynomials $1, (z-\lambda_i), ...(z-\lambda_i)^{h-1}$ are basis for the polynomials $p(z), deg_z p(z) \leq h-1$. write:

$$(z-\lambda_i)^h = (z-\lambda_i)^h - z^h + z^h - \lambda_i^h + \lambda_i^h.$$

=

$$\sum_{i=0}^{v=h-1} c_v (z-\lambda_i)^v + z^h - \lambda_i^h$$

Thus the $detB_l$ is equal to the determinant of a matrix where the last column is replaced by $\int_{a_i} (z^{d(l)} - \lambda_i^{d(l)} \times \frac{dz}{s_l}$. Hence $detB_l = detD_l - detE_l$ where the last columns of D_l, E_l are replaced by: $z^{d(l)}$ and $\lambda_i^{d(l)}$ respectively. By induction assumption we see that $detD_l = \sum_{v=1}^{d(l)-1} detC_v$. So to finish the proof we need to show that $detE_l = detC_{d_l}$. On the matrix E_l perform the elementary operations by swapping the first and last column. dividing the first column by λ_i^h and multiplying the last column by λ_i^h respectively results in a matrix $E_l^{'}$ where the first $d(l)-1$ columns are $\int_{a_i} (z-\lambda_i)^v$ and the last column is $\lambda_i^{d(l)} \frac{\partial}{\partial \lambda_i} \int_{a_h} \frac{dz}{s_l(z)}$. Perform elementary operations on the first $d(l)-1$ to replace the columns with $\int_{a_i} (z-\lambda_i)^v$. Now we can write :

$$\frac{z^j}{z-\lambda_i} = \sum_{h=1}^{j-1} \lambda_i^{h-1} z^{j-h} + \frac{\lambda_i^j}{z-\lambda_i}.$$

But:

$$\frac{\partial}{\partial \lambda_i} \frac{z^j dz}{s_l} = \{lR_i/N\} z^j \frac{dz}{s_l (z-\lambda_i)}$$

Hence:

$$\frac{\partial}{\partial \lambda_i} \int_{a_k} \frac{z^j dz}{s_l} = \lambda_i^j \times \frac{\partial}{\partial \lambda_i} \int_{a_k} \frac{dz}{s_l} + \sum_{h=1}^{j-1} \lambda_i^{h-1} A_{j-h,i}$$

Set $j = d(l)$ and using the fact (yet again!) that elementary operations don't alter the determinant to conclude the result. To finish the proof of **8.5** observe that

$$\frac{\sum_{h=1}^{N-1} \sum_{j=0}^{d(h)} detC_i}{detC} = \frac{\partial}{\partial \lambda_i} \log(detC).$$

■

Let us define the following object we will work closely when showing Thomae:

Definition 8.7. Let $P=(x,y)\in X$ be a non branch point with a local coordinate z. Define:

$$G_z(z)=\lim_{y\to x}\left[\omega(x,y)-\frac{dz(x)dz(y)}{(z(y)-z(x))^2}\right]$$

Now taking the local coordinate $t=(z-\lambda_i)^{\frac{1}{N}}$ around the branch point λ_i we have the following corollary:

Corollary 8.8. *The coefficient of $t^{N-2}dt^2$ in the expansion of $G_z(z)$ in $t=(z-\lambda_i)^{\frac{1}{N}}$ is:*

$$-N\sum_{j=1,j\neq i}^{m}\frac{\gamma_{ij}}{\lambda_i-\lambda_j}-NlogdetA,$$

where

$$\gamma_{ij}=\sum_{h=0}^{N-1}\{hR_i/N\}\{hR_j/N\}$$

To proceed further we learned the following from [Na] see([F] Corollary 2.12) that if e belongs to the Jacobian such that $\theta[e](0,\tau)\neq 0$ then:

$$S[e](x,y)=\omega(x,y)+\sum_{i,j=1}^{g}\frac{\partial^2\log\theta[e](0)}{\partial z_i\partial z_j}v_i(x)v_j(y),$$

$v_i(x),v_j(x)$ are holomorphic differentials on the surface. Using this we obtain the following expression for $G_z(z)$

Proposition 8.9.

$$G_z(z)=\frac{1}{N}\sum_{i,j=1}^{m}\frac{q(\beta_i,\beta_j)dz(x)^2}{(z(x)-\lambda_i)(z(x)-\lambda_j)}-\sum_{i,j=1}^{g}\frac{\partial^2\log\theta[e](0)}{\partial z_i\partial z_j}v_i(x)v_j(y), \tag{12}$$

Passing to the local coordinate $t=(z-\lambda_i)^{\frac{1}{N}}$ we obtain the following corollary:

Corollary 8.10. *The coefficient of $t^{N-2}dt^2$ in the Laurent expansion of $G_z(z)$ is :*

$$2N\sum_{j=1,j\neq i}^{j=m}\frac{q(\beta_i,\beta_j)}{\lambda_i-\lambda_j}-\frac{1}{(N-2)!}\sum_{r,s=1}^{g}\sum_{\alpha=0}^{N-2}\frac{(N-2)!}{\alpha!(N-\alpha)!}\frac{\partial^2\log\theta[e](0)}{\partial z_i\partial z_j}{v_r}^{\alpha}(P_i){v_s}^{\alpha}(P_i), \tag{13}$$

${v_r}^{\alpha}(P_i)$ is the coefficient of dt in the expansion of $v_r(x)$ in the local coordinate t.

9. Variational formula for the period matrix and Thomae for general cyclic covers

[Na] shows the following formula that can be generalized to any cyclic cover:

Theorem 9.1. *If τ is a period matrix with respect to the fixed homology basis $a_i,b_j,1\leq i,j\leq g$ then*

$$\frac{d\tau_{jk}(0)}{dt}=\frac{1}{N(N-2)!}\sum_{\alpha=0}^{N-2}\left(\begin{array}{c}N-2\\ \alpha\end{array}\right){v_j}^{\alpha}(Q_i)\,v_k^{N-2-\alpha}(Q_i) \tag{14}$$

Now let us show Thomae formula. As in [Na] we write the logarithmic derivative of the theta function on the divisor: $e_{\overline{\beta}} = u(\beta_i P_i) - K_{z_0} - u(\sum_{i=1}^{N\infty_i})$

$$\frac{\partial \log\theta\left[e_{\overline{\beta}}\right]}{\partial\lambda_i}(0,\tau) = \frac{d}{dt}\log\theta_t\left[e_{\overline{\beta}}\right](0)|_{t=0}(0,\tau)$$

By the chain rule the last expression is:

$$\frac{d}{dt}\log\theta_t\left[e_{\overline{\beta}}\right](0)|_{t=0}(0,\tau) = \sum_{1\le k,r\le g}\frac{\partial \log\theta\left[e_{\overline{\beta}}\right]}{\partial\tau_{kr}}(0)\frac{d\tau_{kr}}{dt}$$

Now use the heat equation to rewrite the last expression as:

$$\frac{1}{2}\sum_{1\le k,r\le g}\frac{1}{\theta\left[e_{\overline{\beta}}\right](0,\tau)}\frac{\partial\theta\left[e_{\overline{\beta}}\right]}{\partial z_k\partial z_r}(0)\frac{d\tau_{kr}}{dt}$$

We showed in (**7.6**)

$$\frac{\partial\theta[e]}{\partial z_i}(0) = 0$$

The last sum equals:

$$\frac{1}{2}\sum_{1\le k,r\le g}\frac{\partial \log\theta\left[e_{\overline{\beta}}\right]}{\partial z_k\partial z_r}(0)\frac{d\tau_{kr}}{dt}$$

Now use the theorem **9.1**, **13** and **8.8** to conclude that:

$$\frac{\partial \log\theta\left[e_{\overline{\beta}}\right]}{\partial\lambda_i}(0,\tau) = \frac{1}{2}\frac{\partial}{\partial\lambda_i}\log detA + \sum_{j=1,j\neq i}^{m}\frac{q(\beta_i,\beta_j)}{\lambda_i-\lambda_j} + \frac{1}{2}\sum_{j=1,j\neq i}^{j=m}\frac{\gamma_{ij}}{\lambda_i-\lambda_j}$$

Integrate the system of first order differential equations to get the following theorem:

Theorem 9.2. *Let β_i be integer numbers and $0 \le \beta_i \le N-1$. such that*

$$\sum_{i=0}^{m}\overline{\beta_i + kR_i} = \frac{r}{2}.$$

Then there is a complex number C such that:

$$\theta\left[\beta_i P_i - K_{z_0} - \sum_{i=1}^{N}\infty_i\right] = CdetA \times \prod_{i,j=1..m,i\neq j}(\lambda_i-\lambda_j)^{\beta_{ij}+\gamma_{ij}} \tag{15}$$

where $\beta_{lj} = \sum_{k=0}^{N-1}\{\beta_l + kR_l\}\{\beta_j + kR_j\}$ and $\gamma_{lj} = \sum_{w=0}^{N-1}\{wR_i\}\{wR_j\}$.

References

[Ac1] R.Accola *On Cyclic Trigonal Riemann Surfaces* Transactions of The American Mathematical Society Vol.283 pp.423-449

[Ac2] R.Accola *Riemann Surfaces, Theta Functions, and Abelian Automorphism Groups* Lecture Notes in Mathematics vol.483 Springer Verlag

[BR] Bershadsky and Radul *Fermionic Fields on Z_n curves* Communications in Mathematical Physics vol.116 pp 689-700 (1988)

[BR1] Bershadsky and Radul*Conformal Field Theories with additional Z_N symmetry*

Internarnational Journal of Modern Physics A vol.2 pp 165-178 (1987)

[EF] D.Ebin, H.Farkas Thomae Formula for Z_N curves to appear in Journal D'Analyse [Fa] J. Fay *Theta functions on Riemann Surfaces* Lecture Notes in Mathematics Vol. 352, 1973

[FK] H.Farkas,I.Kra *Riemann Surfaces* Graduate Text in Mathematics 1992, Springer Verlag

[Fr] M.Fried Private communications that took place in March 2009

[FK] H.Farkas and I.Kra *Riemann Surfaces* Second Edition, Springer Verlag 1992

[GG] Gabino Gonsalez Dies *Non Special divisors supported on the branch set of p-gonal Riemann Surfaces* London Mathematical Society Lecture Note Series vol.368, 2009

[Na] A. Nakayashiki, On the Thomae formula for Z_N curves *Publ. Res. Inst. Math Sci.* **33** (6)(1997) 987-1015