On the Connectedness of the Branch Loci of Moduli Spaces of Riemann Surfaces I

Milagros Izquierdo

Joint work with A. Costa and Gabriel Bartolini
State College, 24 October 2009

Given a surface X of genus $g \geq 2$ The equivalence:

(X, complex atlas)
($\mathcal{C}_X$, birational structure)
$X \equiv \frac{\mathcal{H}}{\Delta}$, with Δ a (cocompact) Fuchsian group
Δ (cocompact) discrete subgroup of $PSL(2, \mathbb{R})$

Surface Fuchsian Group $\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid \Pi[a_i, b_i] = 1 \rangle$
Teichmüller space $\mathcal{T}_g$, space of geometries on a surface of genus g
$\mathcal{T}_g = \{\sigma : \Gamma_g \to PSL(2,\mathbb{R}) \mid \sigma\, injective,\ \sigma(\Gamma_g) \text{ discrete } \}/PSL(2,\mathbb{R})$
Moduli space $\mathcal{M}_g$, space (orbifold) of conformal structures on a surface of genus g
Mapping Class Group (Teichmüller Modular Group)
$M_g = \frac{Diff(X)}{Diff_0(X)} = Out(\Gamma_g)$

Universal Covering $\mathcal{M}_g = \mathcal{T}_g / M_g$
$\mathcal{B}_g$ Branching Locus = Singular Locus of $\mathcal{M}_g$

$\mathcal{B}_g = \{X \in \mathcal{M}_g \mid Aut(X) \neq 1\}$
$g = 2$ $\mathcal{B}_2 = \{X \in \mathcal{M}_g \mid Aut(X) \rangle C_2 = \langle h \rangle\}$
(h hyperelliptic involution)

$g = 1$ Euclidean case: $\mathcal{T}_1 = \mathcal{H}, M_1 = PSL(2,\mathbb{Z}), \mathcal{B}_1 = \{i, e^{i\pi/3}\}$,

Δ (cocompact) discrete subgroup of $PSL(2, \mathbb{R})$
Δ acts properly discontinuously on $\mathcal{H}$
A (compact) Riemann Surface of genus $g \geq 2$ $\quad X = \frac{\mathcal{H}}{\Delta}$
Δ has presentation:
generators: $x_1, ..., x_r, a_1, b_1, ..., a_h, b_h$
relations: $x_i^{m_i}, i = 1 : r$, $x_1...x_r a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$
x_i: generator of the maximal cyclic subgroups of Δ
$X = \frac{\mathcal{H}}{\Delta}$: orbifold with r cone points and underlying surface of genus g
Algebraic structure of Δ and geometric structure of X are determined by the signature $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
Area of Δ: area of a fundamental region P
$\mu(\Delta) = 2\pi(2h - 2 + \sum_1^r (1 - \frac{1}{m_i}))$
X hyperbolic equivalent to $P/\langle\text{pairing}\rangle$

G finite group of automorphisms of $X_g = \mathcal{H}/\Gamma_g$, Γ_g a surface Fuchsian group iif there exist
Δ Fuchsian group and epimorphism $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma_g$
θ is the monodromy of the covering $f : \mathcal{H}/\Gamma_g \to \mathcal{H}/\Delta$

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & \swarrow & & & \\ X/ = \mathcal{H}/\Gamma_g & & \downarrow & & \Delta\text{: lifting to } \mathcal{H} \text{ of } G \\ & \searrow & & & \\ & & X/G = \mathcal{H}/\Delta & & \end{array}$$

An automorphism of X_g will fix the class of the uniformizing Fuchsian group

A morphism $f : X = \mathcal{H}/\Lambda \to Y = \mathcal{H}/\Delta$, X, Y Riemann surfaces, group inclusion $i : \Lambda \to \Delta$
Covering f determined by monodromy $\theta : \Delta \to \Sigma_N$,
$\Lambda = \theta^{-1}(STb(1))$
(symbol $\leftrightarrow$ Λ-coset $\leftrightarrow$ sheet for f)

Theorem (Singerman 1971) Λ (and so i) determined θ (and Δ): If $s(\Delta) = (h; m_1, \ldots, m_r)$, then
$s(\Lambda) = (h'; m'_{11}, \ldots, m'_{1s_1}, \ldots, m'_{r1}, \ldots, m'_{rs_r})$ iff $\theta : \Delta \to \Sigma_{|\Delta:\Lambda|}$ s.t.
i) Riemann-Hurwitz $\frac{\mu(\Lambda)}{\mu(\Delta)} = |\Delta : \Lambda|$
ii) $\theta(x_i)$ product of s_i cycles each of length $\frac{m_i}{m'_{i1}}, \ldots, \frac{m_i}{m'_{is_i}}$

Δ abstract Fuchsian group $\quad s(\Delta) = (h; m_1, \ldots, m_r)$
$\mathcal{T}_\Delta = \{\sigma : \Delta \to PSL(2,\mathbb{R}) \,|\, \sigma \mathit{injective}, \, \sigma(\Gamma_g) \text{ discrete } \}/PSL(2,\mathbb{R})$
An element in $PSL(2,\mathbb{R})$ acts by conjugation
Teichmüller space $\mathcal{T}_\Delta$ has a complex structure of dim $3h-3+r$, difeomorphic to a ball of dim $6h-6+2r$.

If Λ subgroup of Δ $(i : \Lambda \to \Delta) \Rightarrow i_* : \mathcal{T}_\Delta \to \mathcal{T}_\Lambda$ embedding
Γ_g surface Fuchsian group $\quad \Gamma_g \leq \Delta \quad \mathcal{T}_\Delta \subset \mathcal{T}_{\Gamma_g} = \mathcal{T}_g$
G finite group $\quad \mathcal{T}_g^G = \{[\sigma] \in \mathcal{T}_g \,|\, g[\sigma] = [\sigma] \, \forall g \in G\} \neq \emptyset$
$\mathcal{T}_g^G$: surfaces with G as a group of automorphisms.

Moduli spaces

Consider two marked surfaces in $\mathcal{T}_\Delta$ and the diagram

$$\begin{array}{ccccc} \mathcal{H}/\sigma_1(\Delta) & = & X_1 & \nwarrow [\sigma_1] & \\ \text{biholomorphic} \quad \sigma & & \downarrow & & \mathcal{H}/\Delta \quad \text{The moduli space} \\ \mathcal{H}/\sigma_2(\Delta) & = & X_2 & \swarrow [\sigma_2] & \end{array}$$

$\mathcal{M}_\Delta$ is the quotient space when X_1, X_2 conformally equivalent, i.e. $[\sigma_2 \cdot \sigma_1^{-1}]$ conformal

Mapping class group $M(\Delta) = Out(\Delta) = \frac{Diff(\mathcal{H}/\Delta)}{Diff_0(\mathcal{H}/\Delta)}$

$\Delta = \pi_1(\mathcal{H}/\Delta)$ as orfibold

$M(\Delta)$ acts properly discontinuously on $\mathcal{T}_\Delta$

$\mathcal{M}_\Delta = \mathcal{T}_\Delta / M(\Delta)$

Surfaces with automorphisms : **Branch Locus**

Consider $\mathcal{T}_g$ and $\beta \in M_g$, we have

$$\begin{array}{ccccc} \mathcal{H}/\Delta_g = X & \stackrel{\sigma}{\to} & \sigma(X) & & \\ \downarrow & & \downarrow & & \text{biconformal} \\ \beta_*(X) & \stackrel{\sigma}{\to} & \sigma\beta(X) & & \end{array}$$

$\beta[\sigma] = [\sigma] \quad \Leftrightarrow \quad \gamma \in PSL(2\mathbb{R}), \quad \sigma(\Gamma_g) = \gamma^{-1}\sigma\beta(\Gamma_g)\gamma$

γ induces an automorphism of $[\sigma(X)]$

$Stb_{\mathcal{M}_g}[\sigma] = \{\beta \in M_g \mid \beta[\sigma] = [\sigma]\} = Aut(\sigma(X))$

$G = Aut(X)$ finite, determines a conjugacy class of finite subgroups of M_g, the **symmetry** of X

X_g, Y_g equisymmetric if $Aut(X_g)$ conjugate to $Aut(Y_g)$ ($Aut(X_g)$: full automorphism group. Since actions $\theta : \Delta \to Aut(X_g)\, ker(\theta) = \Gamma_g$, we need Singerman's list of non-maximal signatures. A signature s is called **finitely maximal** if for any Fuchsian group Δ with $s(\Delta) = s$ and a group Δ' containing Δ we have $\dim\mathcal{T}_{\Delta'} < \dim\mathcal{T}_{\Delta}$)

Action: $\theta : \Delta \rightarrow Aut(X_g) = G,\ ker(\theta) = \Gamma_g$
$Aut(X) = G$ conjugate $Aut(Y) \quad \Leftrightarrow \quad w \in Aut(G), h \in Diff(X_0)$
$\epsilon, \epsilon' : G \rightarrow Diff(X_0), \epsilon'(g) = h\epsilon w(g)h^{-1}$
Two (surface) monodromies $\theta_1, \theta_2 : \Delta \rightarrow G$ topologically equiv.

$$\begin{array}{ccccc} & \Delta & \stackrel{\theta_1}{\rightarrow} & G & \\ \text{actions of G} \quad \beta \in Aut(\Delta) & \downarrow & & \downarrow & w \in Aut(G) \\ & \Delta & \stackrel{\theta_2}{\rightarrow} & G & \end{array}$$

$\theta_1,\ \theta_2$ equiv under $\mathcal{B}(\Delta) \times Aut(G)$, $\mathcal{B}(\Delta)$ **braid group**

Broughton (1990): **Equisymmetric Stratification**
$\mathcal{M}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{is} G\}$
$\overline{\mathcal{M}}_g^{G,\theta} = \{X \in \mathcal{M}_g \,|\, \text{symmetry type of } X \text{contains} G\}$
$\mathcal{M}_g^{G,\theta}$ smooth, connected, locally closed al. subvar. of $\mathcal{M}_g$, dense in $\overline{\mathcal{M}}_g^{G,\theta}$ $\qquad \mathcal{B}_g = \cup \mathcal{M}_g^{G,\theta}$
Costa-I (2008) $\mathcal{B}_g = \cup \mathcal{M}_g^{C_p,\theta}$ (Cornalba 1987-2008)
We need to look at maximal actions of C_p

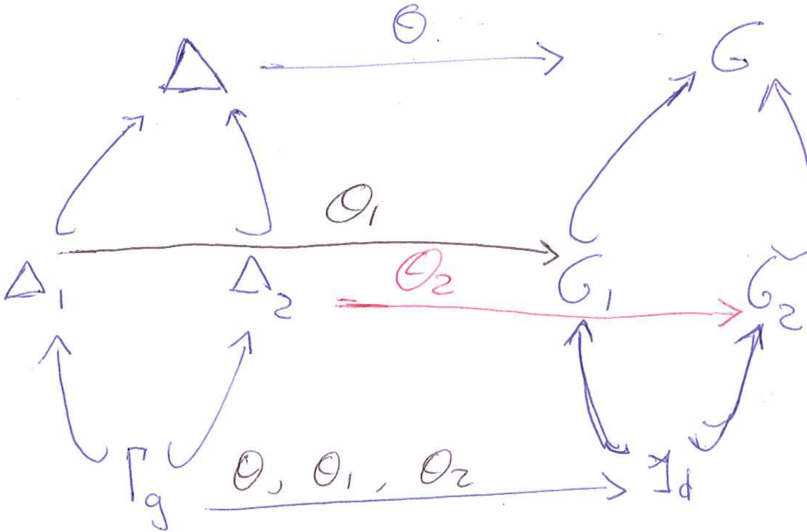
Costa-I (2007) Equisymmetric Stratification of $\mathcal{B}_4$
Bartolini-I (2009) Equisymmetric Stratification of $\mathcal{B}_5$

Connectedness, we are interested in $Y \in \overline{\mathcal{M}}_g^{G_1,\theta_1} \cap \overline{\mathcal{M}}_g^{G_2,\theta_2}$
Finding $\theta : \Delta \to G = Aut(Y)$ extends both $\theta_1 : \Delta_1 \to G_1$ and $\theta_2 : \Delta_2 \to G_2$ with $Ker(\theta) = Ker(\theta_1) = Ker(\theta_2) = \Gamma_g$

Geometrically

$$\begin{array}{ccccc} & f_{\theta_1} & Y = \mathcal{H}/\Gamma_g & f_{\theta_2} & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{H}/\Delta_1 & & f_\theta & & \mathcal{H}/\Delta_2 \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{H}/\Delta & & \end{array}$$

$G_1 = C_{p_1}$ and $G_2 = C_{p_2}$


Θ
Δ
G
Θ1
Δ1
Δ2
Θ2
G1
G2
Γg
Θ, Θ1, Θ2
Id

Costa-I (2008) $\mathcal{B}_4$ is connected

$\mathcal{B}_4 = \overline{\mathcal{M}}_4^{C_2,0} \cup \overline{\mathcal{M}}_4^{C_2,1} \cup \overline{\mathcal{M}}_4^{C_2,2} \cup \overline{\mathcal{M}}_4^{C_3,01} \cup \overline{\mathcal{M}}_4^{C_3,02} \cup \overline{\mathcal{M}}_4^{C_3,1} \cup \overline{\mathcal{M}}_4^{C_5,1}$

$\overline{\mathcal{M}}_4^{C_2,0}$ hyperelliptic surfaces, $\overline{\mathcal{M}}_4^{C_2,1}$ elliptic-hyperelliptic surfaces

$\overline{\mathcal{M}}_4^{C_3,01}$ cyclic trigonal surfaces, stabilazers rotate in opposite directions

$\overline{\mathcal{M}}_4^{C_3,02}$ cyclic trigonal surfaces, all stabilazers rotate the same direction

$\overline{\mathcal{M}}_4^{C_5,1}$ cyclic pentagonal surfaces with $Aut(X) = C_5$

$\overline{\mathcal{M}}_4^{C_5,2} = \overline{\mathcal{M}}_4^{D_{10}} \subset \overline{\mathcal{M}}_4^{C_2,0} \cap \overline{\mathcal{M}}_4^{C_2,2}$

$\overline{\mathcal{M}}_4^{D_3 \times D_3} \subset \overline{\mathcal{M}}_4^{C_3,01} \cap \overline{\mathcal{M}}_4^{C_3,2} \subset \overline{\mathcal{M}}_4^{C_3,01} \cap \overline{\mathcal{M}}_4^{C_2,1}$

$\overline{\mathcal{M}}_4^{C_3 \times D_3} \subset \overline{\mathcal{M}}_4^{C_3,1} \cap \overline{\mathcal{M}}_4^{C_3,02} \cap \overline{\mathcal{M}}_4^{C_2,1}$

$\overline{\mathcal{M}}_4^{C_6 \times C_2} \subset \overline{\mathcal{M}}_4^{C_3,02} \cap \overline{\mathcal{M}}_4^{C_2,2} \cap \overline{\mathcal{M}}_4^{C_2,1}$

$T_4 = \overline{\mathcal{M}}_4^{C_{15}} \in \overline{\mathcal{M}}_4^{C_3,02} \cap \overline{\mathcal{M}}_4^{C_5,1}$

Kulkarni (1991) . Existence of isolated points in $\mathcal{B}_g$ iff $g = 2$ or 2g+1 a prime ≥ 11 (Cornalba 1987-2008)

Isolated points are given by actions

$\theta : \Delta(0; p, p, p) \to C_p,\ p = 2g + 1$

The actions of C_7 in $\mathcal{M}_3$ extend to actions of C_{14} or $PSL(2, 7)$

Bartolini-I (2009): $\overline{\mathcal{M}}_g^{C_2}$ and $\overline{\mathcal{M}}_g^{C_3}$ belong to the same connected component of $\mathcal{B}_g$

Costa-I (2009): $\mathcal{B}_g$ contains isolated strata of dimension 1 iff g+1 is a prime ≥ 11

Actions $\theta : \Delta(0; p, p, p, p) \rightarrow C_p, \quad p = g + 1$

i) Actions $\theta : \Delta \rightarrow C_p$ with $\theta(x_1) = \theta(x_2) = a, \theta(x_3) = \theta(x_4) = a^{-1}$ non-maximal. Full automorphism group is D_{2p}. $\overline{\mathcal{M}}_g^{C_p,2i} \subset \overline{\mathcal{M}}_g^{C_2,g/2} \cap \overline{\mathcal{M}}_g^{C_2,0}$

ii) Actions $\theta_i : \Delta \rightarrow C_p$ with $\theta_i(x_1) = \theta_i(x_2) = \theta_i(x_3) = a$.
The surface $X_g^{\varphi_i}$ induced by $\varphi_i : \Lambda(0; p, 2p, 2p) \rightarrow C_{2p}$, $X_g^{\varphi_i} \in \overline{\mathcal{M}}_g^{C_p,2i} \cap \overline{\mathcal{M}}_g^{C_2,g/2}$

iii) Actions $\theta_i : \Delta \rightarrow C_p$ with $\theta_i(x_1) = \theta_i(x_2) = a, \theta_i(x_3) = a^i$.
The surface X_g^{ϕ} induced by $\phi : \Lambda(0; 3, p, 3p) \rightarrow C_{3p} = \langle a, t \rangle$, $X_g^{\phi} \in \overline{\mathcal{M}}_g^{C_p,3} \cap \overline{\mathcal{M}}_g^{C_3,0}$

iv) Actions $\theta_i : \Delta \to C_p$ with
$\theta_i(x_1) = a, \theta_i(x_2) = a^{-1}, \theta_i(x_3) = a^i, \theta_i(x_4) = a^{-i}$ non-maximal.
Full automorphism group is D_p. The strata $\overline{\mathcal{M}}_g^{C_p,4i} \subset \overline{\mathcal{M}}_g^{C_2,g/2}$

v) Actions $\theta_i : \Delta(0; p, p, p, p) \to C_p$ with $\theta_i(x_1) = a, \theta_i(x_2) = a^i, \theta_i(x_3) = a^j, \quad i \neq 1, p-1, j \neq 1, p-1, i, i-1.$
This case does not exist for $p = 5$ and $p = 7$
This action is maximal and the strata contain no curve with more symmetry.

These strata are isolated