# Wilson's operations on regular dessins and cyclotomic fields of definition

Jürgen Wolfart

joint work with Gareth Jones and Manfred Streit,
to appear in the Proceedings of the LMS

Mathematisches Seminar, Goethe Universität Frankfurt

24th October 2009

# Outline

1. Galois invariants of dessins d'enfants
2. Wilson operations
3. Main results
4. Example: $K_n$–dessins

# Dessins d'enfants

Dessins d'enfants (=childrens drawings) are *hypermaps* on compact Riemann surfaces $X$, given in *Walsh representation* (Walsh 1975) as bipartite graphs cutting $X$ into simply connected cells.

They occur in a natural way as preimages of the real interval ○——● between 0 and 1 under Belyĭ functions

$$\beta : X \to \hat{\mathbb{C}}$$

(non–constant, meromorphic and ramified at most above $0, 1, \infty$).

Such functions exist if and only if – as an algebraic curve – $X$ can be defined over a number field (Belyĭ 1979).

# Dessins d'enfants

Dessins d'enfants (=childrens drawings) are *hypermaps* on compact Riemann surfaces $X$, given in *Walsh representation* (Walsh 1975) as bipartite graphs cutting $X$ into simply connected cells.

They occur in a natural way as preimages of the real interval ○——● between 0 and 1 under Belyĭ functions

$$\beta : X \to \hat{\mathbb{C}}$$

(non–constant, meromorphic and ramified at most above $0, 1, \infty$).

Such functions exist if and only if – as an algebraic curve – $X$ can be defined over a number field (Belyĭ 1979).

# Dessins d'enfants

Dessins d'enfants (=childrens drawings) are *hypermaps* on compact Riemann surfaces $X$, given in *Walsh representation* (Walsh 1975) as bipartite graphs cutting $X$ into simply connected cells.

They occur in a natural way as preimages of the real interval ○——● between 0 and 1 under Belyĭ functions

$$\beta : X \to \hat{\mathbb{C}}$$

(non–constant, meromorphic and ramified at most above $0, 1, \infty$).

Such functions exist if and only if – as an algebraic curve – $X$ can be defined over a number field (Belyĭ 1979).

# Galois action

Suppose $X$ is such a smooth projective algebraic curve defined over a number field, and let $\sigma$ be any algebraic conjugation, i.e. an element of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$.

By the action on the coefficients of the defining equation(s) of $X$ and its Belyĭ function, $X$ becomes a new algebraic curve $X^\sigma$ with Belyĭ function $\beta^\sigma$, hence with another dessin.

If $X$ and $\beta$ are defined over the field $k$, the dessin $\mathcal{D}$ is invariant under the action of each $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/k$. For more general $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$, which properties and in particular which invariants has the Galois operation

$$\mathcal{D} \mapsto \mathcal{D}^\sigma \quad ?$$

# Galois action

Suppose $X$ is such a smooth projective algebraic curve defined over a number field, and let $\sigma$ be any algebraic conjugation, i.e. an element of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$.

By the action on the coefficients of the defining equation(s) of $X$ and its Belyĭ function, $X$ becomes a new algebraic curve $X^\sigma$ with Belyĭ function $\beta^\sigma$, hence with another dessin.

If $X$ and $\beta$ are defined over the field $k$, the dessin $\mathcal{D}$ is invariant under the action of each $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/k$. For more general $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$, which properties and in particular which invariants has the Galois operation

$$\mathcal{D} \mapsto \mathcal{D}^\sigma \quad ?$$

# Galois action

Suppose $X$ is such a smooth projective algebraic curve defined over a number field, and let $\sigma$ be any algebraic conjugation, i.e. an element of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$.

By the action on the coefficients of the defining equation(s) of $X$ and its Belyĭ function, $X$ becomes a new algebraic curve $X^\sigma$ with Belyĭ function $\beta^\sigma$, hence with another dessin.

If $X$ and $\beta$ are defined over the field $k$, the dessin $\mathcal{D}$ is invariant under the action of each $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/k$. For more general $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$, which properties and in particular which invariants has the Galois operation

$$\mathcal{D} \mapsto \mathcal{D}^\sigma \quad ?$$

# Galois invariants

- The list of valencies of white vertices = zero orders of $\beta$,
- the list of valencies of black vertices,
- the list of valencies of faces,
- the number of edges = degree of $\beta$,
- the genus of $X$,
- the automorphism group (up to isomorphism, of course),
- regularity of the dessin,
- its type.

# Regularity, type and canonical generators

A dessin is $\mathcal{D}$ is called regular if its (orientation preserving $\Rightarrow$ conformal) automorphism group $\mathrm{Aut}\,\mathcal{D}$ acts transitively on the edges of $\mathcal{D}$. If so, $\beta$ is a normal (ramified) covering map having $\mathrm{Aut}\,\mathcal{D}$ as covering group.

Moreover, all white vertices have the same valency $p$, all black vertices have the same valency $q$, and all faces have the same valency $2r$, and we call $(p, q, r)$ the type of $\mathcal{D}$.

For every edge $e$ of $\mathcal{D}$ one has two canonical generators $a$ and $b$ of $\mathrm{Aut}\,\mathcal{D}$ with the properties

$$a^p = b^q = (ab)^r = 1$$

# Regularity, type and canonical generators

A dessin is $\mathcal{D}$ is called regular if its (orientation preserving $\Rightarrow$ conformal) automorphism group $\mathrm{Aut}\,\mathcal{D}$ acts transitively on the edges of $\mathcal{D}$. If so, $\beta$ is a normal (ramified) covering map having $\mathrm{Aut}\,\mathcal{D}$ as covering group.

Moreover, all white vertices have the same valency $p$, all black vertices have the same valency $q$, and all faces have the same valency $2r$, and we call $(p, q, r)$ the type of $\mathcal{D}$.

For every edge $e$ of $\mathcal{D}$ one has two canonical generators $a$ and $b$ of $\mathrm{Aut}\,\mathcal{D}$ with the properties

$$a^p = b^q = (ab)^r = 1$$

# Regularity, type and canonical generators

A dessin is $\mathcal{D}$ is called regular if its (orientation preserving $\Rightarrow$ conformal) automorphism group $\mathrm{Aut}\,\mathcal{D}$ acts transitively on the edges of $\mathcal{D}$. If so, $\beta$ is a normal (ramified) covering map having $\mathrm{Aut}\,\mathcal{D}$ as covering group.

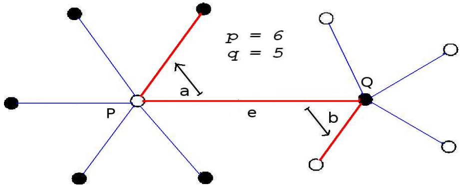
Moreover, all white vertices have the same valency $p$, all black vertices have the same valency $q$, and all faces have the same valency $2r$, and we call $(p, q, r)$ the type of $\mathcal{D}$.

For every edge $e$ of $\mathcal{D}$ one has two canonical generators $a$ and $b$ of $\mathrm{Aut}\,\mathcal{D}$ with the properties

$$a^p = b^q = (ab)^r = 1$$

# Local behaviour of the generators. Multipliers

The generators $a$ (and $b$) fix the white (and black) end vertex $P$ (resp.$Q$) of the edge $e$ and map $e$ to the anticlockwise next one.



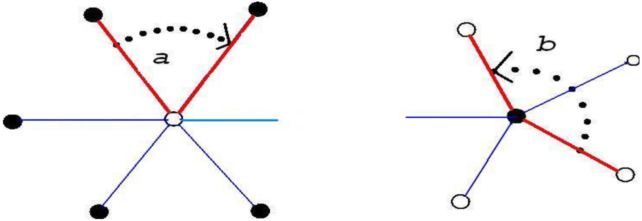

In suitable local coordinates, they look around $P$ (resp. $Q$) like

$$a : z \mapsto e^{2\pi i/p} z \qquad \text{and} \qquad b : w \mapsto e^{2\pi i/q} w$$

We call these roots of unity $\zeta_p$ resp. $\zeta_q$ the multipliers of $a$ in $P$ and $b$ in $Q$.

# Galois action on multipliers

Less easy to prove than to understand: given $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$, a regular dessin $\mathcal{D}$ with canonical generators $a, b$ of $\mathrm{Aut}\,\mathcal{D}$, fixing the neighbour vertices $P, Q$, their Galois conjugates $a^\sigma, b^\sigma$ fix the vertices $P^\sigma, Q^\sigma$ of $\mathcal{D}^\sigma$, but now with multipliers $\sigma(\zeta_p), \sigma(\zeta_q)$, so in general $a^\sigma, b^\sigma$ are no longer canonical generators of $\mathrm{Aut}\,\mathcal{D}$.




The idea is also known under the name *cyclotomic character* of the Galois group (Ihara 1994?)

# Graph isomorphism?

Caution. In general, $P^\sigma$ and $Q^\sigma$ are no longer neighbour vertices, and the underlying graphs for $\mathcal{D}$ and $\mathcal{D}^\sigma$ are not necessarily isomorphic!

Easiest known example: the three regular dessins of type $(2,3,7)$ on the Hurwitz curves of genus 14 with automorphism group $\mathrm{PSL}_2\mathbf{F}_{13}$ are defined over $\mathbf{Q}(\cos\pi/7)$ and Galois conjugate (Streit 2000) but their graphs are not isomorphic.

However: if one neighbouring pair of vertices of the regular dessin $\mathcal{D}$ is mapped under $\sigma$ onto a neighbour pair, then $\sigma$ induces a graph isomorphism.

# Graph isomorphism?

Caution. In general, $P^\sigma$ and $Q^\sigma$ are no longer neighbour vertices, and the underlying graphs for $\mathcal{D}$ and $\mathcal{D}^\sigma$ are not necessarily isomorphic!

Easiest known example: the three regular dessins of type $(2,3,7)$ on the Hurwitz curves of genus 14 with automorphism group $\mathrm{PSL}_2\mathbf{F}_{13}$ are defined over $\mathbf{Q}(\cos\pi/7)$ and Galois conjugate (Streit 2000) but their graphs are not isomorphic.

However: if one neighbouring pair of vertices of the regular dessin $\mathcal{D}$ is mapped under $\sigma$ onto a neighbour pair, then $\sigma$ induces a graph isomorphism.

# Graph isomorphism?

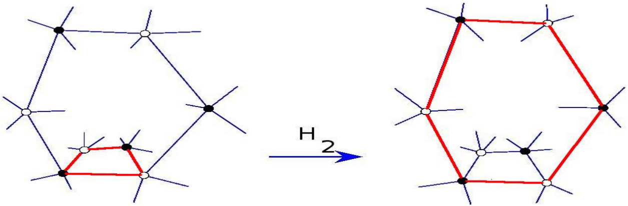
Caution. In general, $P^\sigma$ and $Q^\sigma$ are no longer neighbour vertices, and the underlying graphs for $\mathcal{D}$ and $\mathcal{D}^\sigma$ are not necessarily isomorphic!

Easiest known example: the three regular dessins of type $(2, 3, 7)$ on the Hurwitz curves of genus 14 with automorphism group $\mathrm{PSL}_2\mathbf{F}_{13}$ are defined over $\mathbf{Q}(\cos\pi/7)$ and Galois conjugate (Streit 2000) but their graphs are not isomorphic.

However: if one neighbouring pair of vertices of the regular dessin $\mathcal{D}$ is mapped under $\sigma$ onto a neighbour pair, then $\sigma$ induces a graph isomorphism.

# The topological meaning of the Wilson operations

Let $\mathcal{D}$ be a regular dessin of type $(p, q, r)$, and let $j \in \mathbf{Z}$ be coprime to $pq$. The Wilson operation $H_j$ sends $\mathcal{D}$ to a new regular dessin $H_j\mathcal{D}$ with the same edges but whose faces are bounded by new edge cycles as follows.




Arriving on an edge in its endpoint $P$ one continues not with the next edge in counterclockwise order but with the $j$–th edge in counterclockwise order (Steve Wilson 1979). In particular, $H_{-1}\mathcal{D}$ gives the complex conjugate of the original dessin (Galois conjugate to $\mathcal{D}$, too).

# Invariants and non–invariants of Wilson operators

- The underlying graph of $\mathcal{D}$,
- regularity (see next slide),
- the automorphism group $\mathrm{Aut}\,\mathcal{D}$,
- the valencies $p$ and $q$,
- not necessarily the valency $r$ of the faces,
- hence not necessarily the genus

Examples for the last items: already the dodecahedron map (Wilson 1979) or the regular $(7, 2, 3)$ dessin on Klein's quartic.

# Algebraic definition of Wilson operators

Suppose $\mathcal{D}$ to be a regular dessin with canonical generators $a, b$ of $G := \mathrm{Aut}\,\mathcal{D}$, and $j \in \mathbf{Z}$ coprime to $pq$.

Replace the canonical generators $a, b$ of $G$ with the generators $a^j, b^j$ and do the same with all pairs of canonical generators of $G$ (conjugate to $(a, b)$, of course). These define $H_j\mathcal{D}$ uniquely, now with $a^j, b^j$ as canonical generators of $G = \mathrm{Aut}\,H_j\mathcal{D}$.

# Side remarks, generalizations

This definition makes more sense for the cartographic group of $\mathcal{D}$ (monodromy group of its Belyĭ function) and can be extended in several ways:

- admitting $j$ not coprime to $pq$ (Wilson, losing invariance properties!),
- admitting different exponents for $a$ and $b$ (extremely useful, see our paper in Proc. London Math. Soc.),
- application to non–regular dessins.

# From Galois orbits to Wilson orbits

## Theorem

*Let m be the lcm of p and q and* $\mathcal{D}(k), k \in (\mathbf{Z}/m\mathbf{Z})^*$, *a family of regular dessins defined over the cyclotomic field* $\mathbf{Q}(\zeta_m)$ *and Galois conjugate under all*

$$\sigma_k : \zeta_m \mapsto \zeta_m^k \quad , \quad \mathcal{D}(1) \mapsto \mathcal{D}(k) \, .$$

*Suppose moreover that the Galois action on the dessins* $\mathcal{D}(k)$ *preserves adjacency. Then the Galois action is a Wilson operation, more precisely*

$$\mathcal{D}(k) = \mathcal{D}(1)^{\sigma_k} = H_j \mathcal{D}(1) \quad \textit{for} \quad jk \equiv 1 \bmod m \, .$$

# Idea of proof

Identify $G := \mathrm{Aut}\,\mathcal{D}(1)$ with all its $\sigma_k$–images $\mathrm{Aut}\,\mathcal{D}(k)$ and recall that $\sigma_k$ defines an action on the edges, too. Consider a pair of canonical generators $a, b$ of $G$ for two adjacent vertices in $\mathcal{D}$ : their $\sigma_k$–images are in general not the canonical generators for the image vertices in $\mathcal{D}(k)$ .

Their multipliers indicate that the images are $a^k, b^k$ instead.
On the other hand, $a^j, b^j$ (canonical generators of $H_j\mathcal{D}(1)$ with $jk \equiv 1 \bmod m$, hence $\equiv 1 \bmod p$ and $\equiv 1 \bmod q$) are mapped by $\sigma_k$ to the canonical generators $a, b$, so the claim follows.

# Idea of proof

Identify $G := \mathrm{Aut}\,\mathcal{D}(1)$ with all its $\sigma_k$–images $\mathrm{Aut}\,\mathcal{D}(k)$ and recall that $\sigma_k$ defines an action on the edges, too. Consider a pair of canonical generators $a, b$ of $G$ for two adjacent vertices in $\mathcal{D}$ : their $\sigma_k$–images are in general not the canonical generators for the image vertices in $\mathcal{D}(k)$ .

Their multipliers indicate that the images are $a^k, b^k$ instead.
On the other hand, $a^j, b^j$ (canonical generators of $H_j\mathcal{D}(1)$ with $jk \equiv 1 \bmod m$, hence $\equiv 1 \bmod p$ and $\equiv 1 \bmod q$) are mapped by $\sigma_k$ to the canonical generators $a, b$, so the claim follows.

# Generalizations, converse?

The hypothesis concerning adjacency preservation cannot be completely omitted, but weakened using an assumption that for all $k$ the algebraic conjugation

$$\sigma_k : \zeta_m \mapsto \zeta_m^k \qquad \Rightarrow \qquad \mathcal{D} \rightarrow \mathcal{D}^{\sigma_k}$$

induces a graph isomorphism compatible with the automorphism group.

The term compatible has to be made precise!

For applications, the converse of the theorem turns out to be more important.

# Generalizations, converse?

The hypothesis concerning adjacency preservation cannot be completely omitted, but weakened using an assumption that for all $k$ the algebraic conjugation

$$\sigma_k : \zeta_m \mapsto \zeta_m^k \qquad \Rightarrow \qquad \mathcal{D} \to \mathcal{D}^{\sigma_k}$$

induces a graph isomorphism compatible with the automorphism group.

The term compatible has to be made precise!

For applications, the converse of the theorem turns out to be more important.

# From Wilson to cyclotomy

## Theorem

*Let $\mathcal{D}$ be a regular dessin of type $(p, q, r)$, $m$ the lcm of $p$ and $q$. Suppose that all Wilson transforms $H_j\mathcal{D}$, $j \in (\mathbf{Z}/m\mathbf{Z})^*$ are of the same type and form a family invariant under the action of $\mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$.*

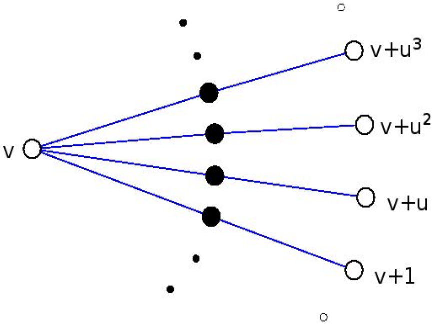
*Then all $H_j\mathcal{D}$ are defined over a subfield $K$ of the cyclotomic field $\mathbf{Q}(\zeta_m)$, forming one Galois orbit.*

*Their (minimal) field of definition $K$ is the fixed field of*

$$\{j \in (\mathbf{Z}/m\mathbf{Z})^* \mid H_j\mathcal{D} \cong \mathcal{D}\} \leq (\mathbf{Z}/m\mathbf{Z})^* \cong \mathrm{Gal}\,\mathbf{Q}(\zeta_m)/\mathbf{Q}\ .$$

# Dessins for $K_n$

Biggs 1971, Lynne James/Gareth Jones 1985 : Regular dessins of type $(n-1, 2, ?)$ based on the complete graphs $K_n$ exist if and only if $n = p^e$ is a prime power. They can be described taking $u$ as a generator of the multiplicative group $\mathbf{F}_n^*$ of the finite field $\mathbf{F}_n$. Label the $n$ white vertices of the dessin with the elements $v \in \mathbf{F}_n$ and arrange the edges around $v$ as follows.



# Isomorphisms and automorphisms

Call the resulting dessins $\mathcal{D}(n,u)$. Two such dessins $\mathcal{D}(n,u)$ and $\mathcal{D}(n',u')$ are isomorphic if and only if $n=n'$ and $u,u'$ belong to the same Galois orbit of $\mathrm{Gal}\,\mathbf{F}_n/\mathbf{F}_p$, i.e. $u'=u^{p^k}$ for some integer $k$.

Therefore, the $\mathcal{D}(n,u)$ form a family with $\varphi(n-1)/e$ non–isomorphic members. Moreover, this family is uniquely determined by the type and

$$\mathrm{Aut}\,\mathcal{D}(n,u)\;\cong\;\mathbf{F}_n \rtimes \mathbf{F}_n^*\,.$$

The family is therefore Galois invariant.

# Isomorphisms and automorphisms

Call the resulting dessins $\mathcal{D}(n,u)$. Two such dessins $\mathcal{D}(n,u)$ and $\mathcal{D}(n',u')$ are isomorphic if and only if $n=n'$ and $u,u'$ belong to the same Galois orbit of $\mathrm{Gal}\,\mathbf{F}_n/\mathbf{F}_p$, i.e. $u'=u^{p^k}$ for some integer $k$.

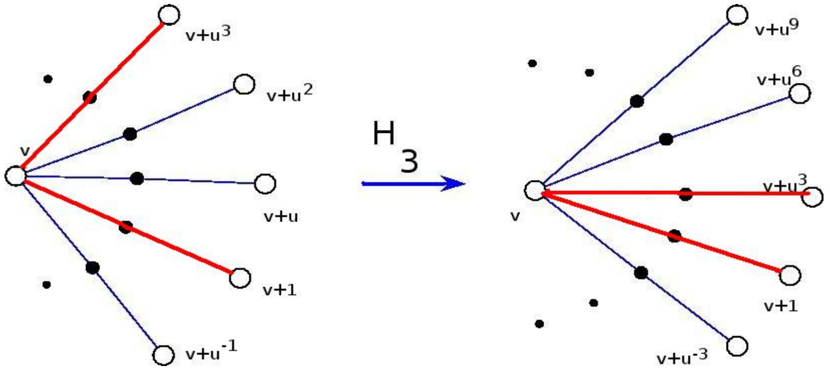
Therefore, the $\mathcal{D}(n,u)$ form a family with $\varphi(n-1)/e$ non–isomorphic members. Moreover, this family is uniquely determined by the type and

$$\mathrm{Aut}\,\mathcal{D}(n,u) \;\cong\; \mathbf{F}_n \rtimes \mathbf{F}_n^* \,.$$

The family is therefore Galois invariant.

# The $K_n$–dessins as Wilson orbit

Moreover, since any two generators $u, u'$ of $\mathbf{F}_n^*$ satisfy $u' = u^j$ for some $j$ coprime to $n-1$, we have $\mathcal{D}(n, u^j) = H_j \mathcal{D}(n, u)$ :



# The field of definition of the $K_n$–dessins

Therefore the Wilson orbit consists of a Galois invariant family of regular dessins, hence we have

**Theorem**

*The $K_n$–dessins $\mathcal{D}(n, u)$ form one Galois orbit and are defined over the splitting subfield of $\mathbf{Q}(\zeta_{n-1})$ for the prime $p$.*

(the largest subfield of $\mathbf{Q}(\zeta_{n-1})$ in which $p$ splits into prime ideals of degree 1, fixed field of the Frobenius automorphism $\zeta_{n-1} \mapsto \zeta_{n-1}^p$.)

# $K_8$, Edmonds maps and Macbeath's curve

The first example for genus $g > 1$ given by this result arises for

$$n = 8 = 2^3\ , \quad (p,q,r) = (7,2,7)\ , \quad g = 7\ , \quad \varphi(n-1)/e = 2\ ,$$

i.e. giving two Galois conjugate dessins defined over a quadratic field of definition $K < \mathbf{Q}(\zeta_7)$; this is $K = \mathbf{Q}(\sqrt{-7})$, so the two dessins are complex conjugate, a chiral pair. They are well known as the Edmonds maps.

However, these two dessins live on the same curve (Singerman 1974), and this curve is in fact well known ($\Leftarrow$ E. Girondo/D. Torres/JW – in preparation – and email communication with Gareth): the Macbeath curve with automorphism group $\mathrm{PSL}_2\mathbf{F}_8$ of order 504, the second smallest Hurwitz group.

# $K_8$, Edmonds maps and Macbeath's curve

The first example for genus $g > 1$ given by this result arises for

$$n = 8 = 2^3 \, , \quad (p,q,r) = (7,2,7) \, , \quad g = 7 \, , \quad \varphi(n-1)/e = 2 \, ,$$

i.e. giving two Galois conjugate dessins defined over a quadratic field of definition $K < \mathbf{Q}(\zeta_7)$; this is $K = \mathbf{Q}(\sqrt{-7})$, so the two dessins are complex conjugate, a chiral pair. They are well known as the Edmonds maps.

However, these two dessins live on the same curve (Singerman 1974), and this curve is in fact well known ($\Leftarrow$ E. Girondo/D. Torres/JW – in preparation – and email communication with Gareth): the Macbeath curve with automorphism group $\mathrm{PSL}_2\mathbf{F}_8$ of order 504, the second smallest Hurwitz group.

# The function field for the Macbeath curve

can be obtained using the function field extensions corresponding to the normal inclusions of the Fuchsian groups

$$\langle 0;7,2,7\rangle \overset{\mathbf{F}_8^*}{\triangleright} \langle 0;2,2,2,2,2,2,2\rangle \overset{C_2}{\triangleright} \left\{ \begin{array}{c} \langle 2;2,2\rangle_1 \\ \langle 2;2,2\rangle_2 \\ \langle 2;2,2\rangle_4 \end{array} \right\} \overset{C_2\times C_2}{\triangleright} \langle 7;-\rangle$$

where the factor group is always given above and the right hand side denotes the surface group of Macbeath's curve.

# Equations for Macbeath's curve

were first given by Macbeath 1965, but result more easily from these function field extensions in an affine space $\mathbf{C}^4$ with coordinates $x, y_1, y_2, y_4$ and three equations of type

$$y_j^2 = (x^7 - 1)/(x - \zeta_7^j) \ , \quad j = 1, 2, 4 \ .$$

The Belyĭ function for one of the Edmonds dessins is given by $(x, y_1, y_2, y_4) \mapsto x^7$.

# Homework

Prove that $\zeta_7 \mapsto \zeta_7^2$ induces an automorphism of this dessin!

What's about the other Edmonds dessin?

And how can we see that all Galois conjugations in $\mathbf{Q}(\zeta_7)$ induce isomorphisms of the curve?

Unsolved problem: there should be a model of Macbeath's curve given by equations defined over $\mathbf{Q}$. Find these equations!

# Homework

Prove that $\zeta_7 \mapsto \zeta_7^2$ induces an automorphism of this dessin!

What's about the other Edmonds dessin?

And how can we see that all Galois conjugations in $\mathbf{Q}(\zeta_7)$ induce isomorphisms of the curve?

Unsolved problem: there should be a model of Macbeath's curve given by equations defined over $\mathbf{Q}$. Find these equations!