Edge-transitive tessellations with non-negative Euler characteristic

Alen Orbanić

Daniel Pellicer

Tomaž Pisanski

Thomas Tucker

Arjana Žitnik

Maps

- **2-CELL EMBEDDING** of a

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- **3-CONECTED GRAPH** on a

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▶ Automorphism $\longrightarrow$ automorphism of the graph that extends to an auto-homomeorphism of the surface.

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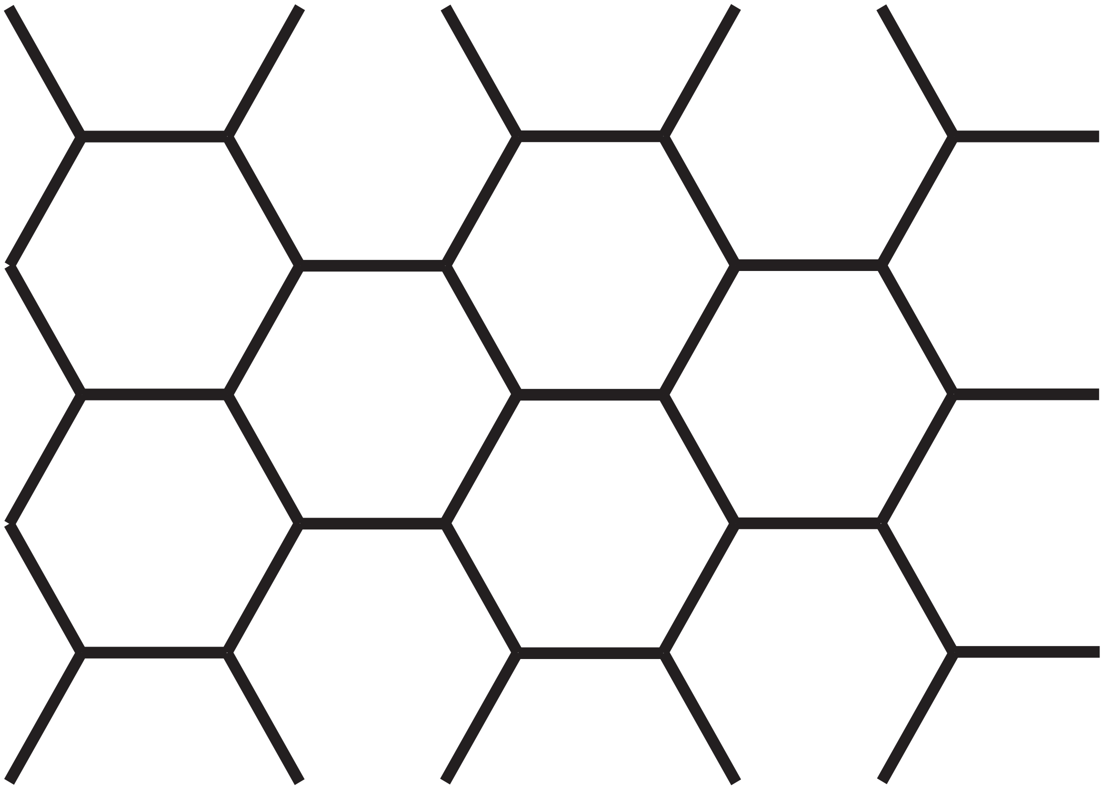
▶ Automorphism $\longrightarrow$ automorphism of the graph that extends to an auto-homomeorphism of the surface.
▶ Flag $\longrightarrow$ triple of incident vertex, edge and cell (face)

Flag graph



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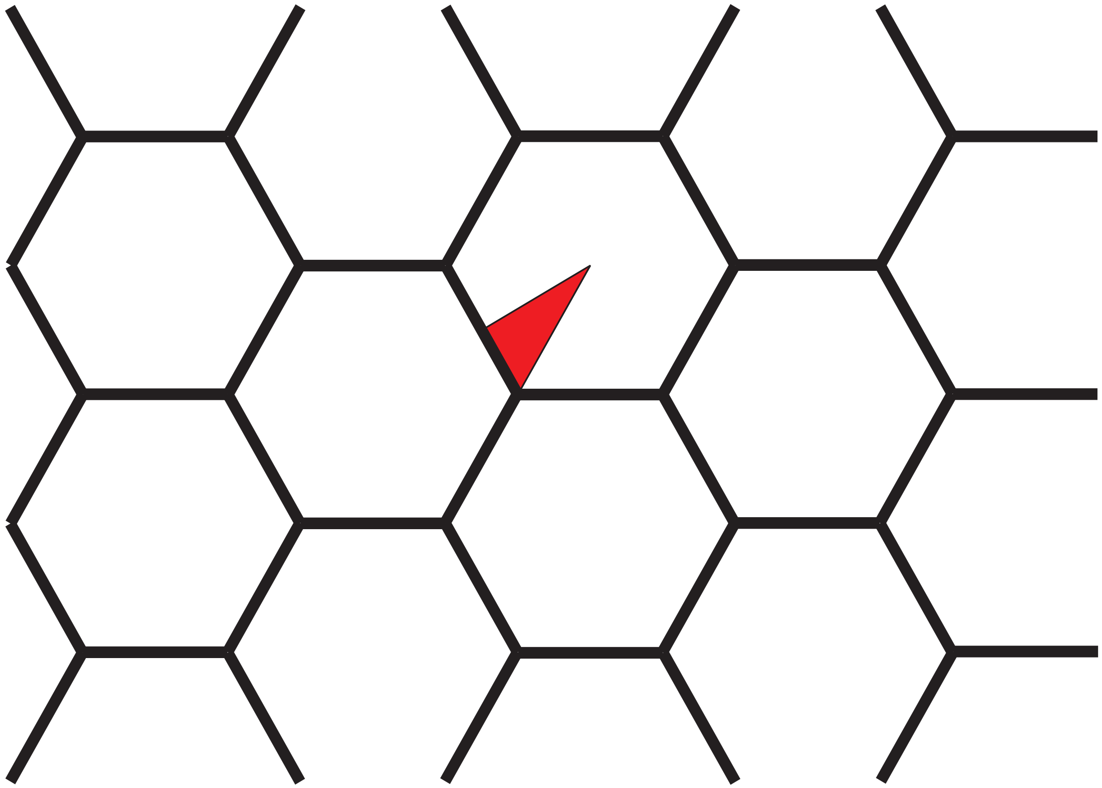

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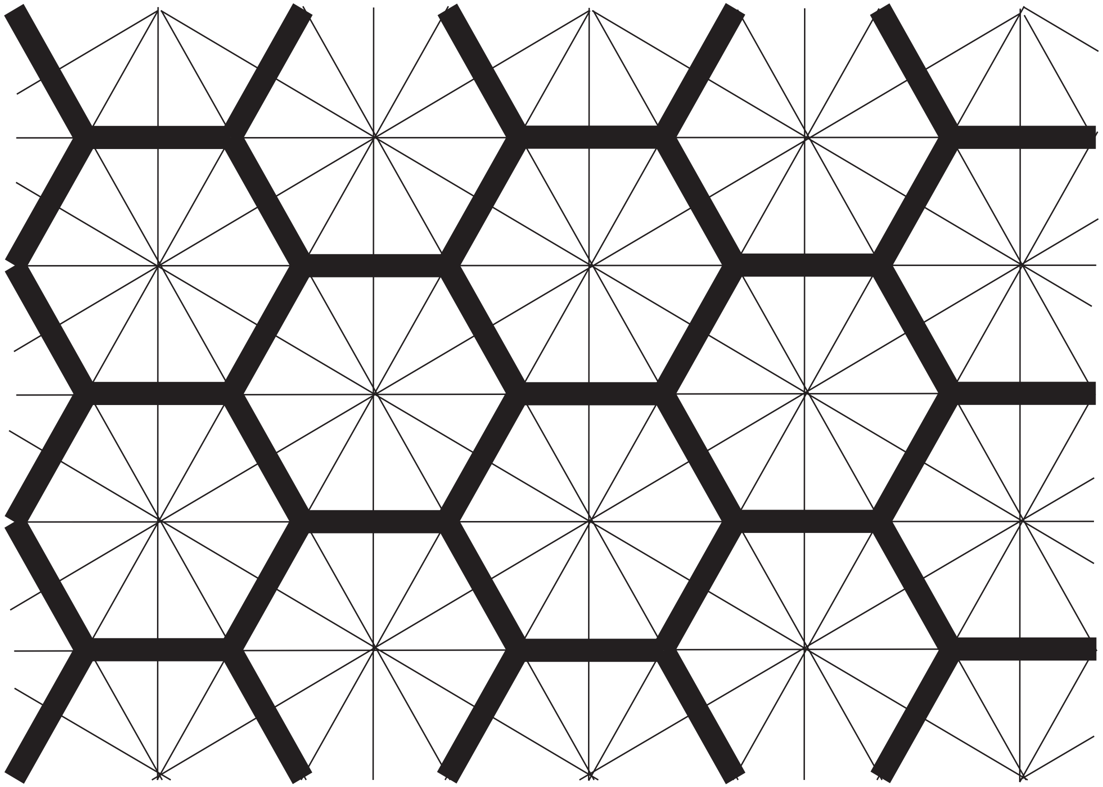

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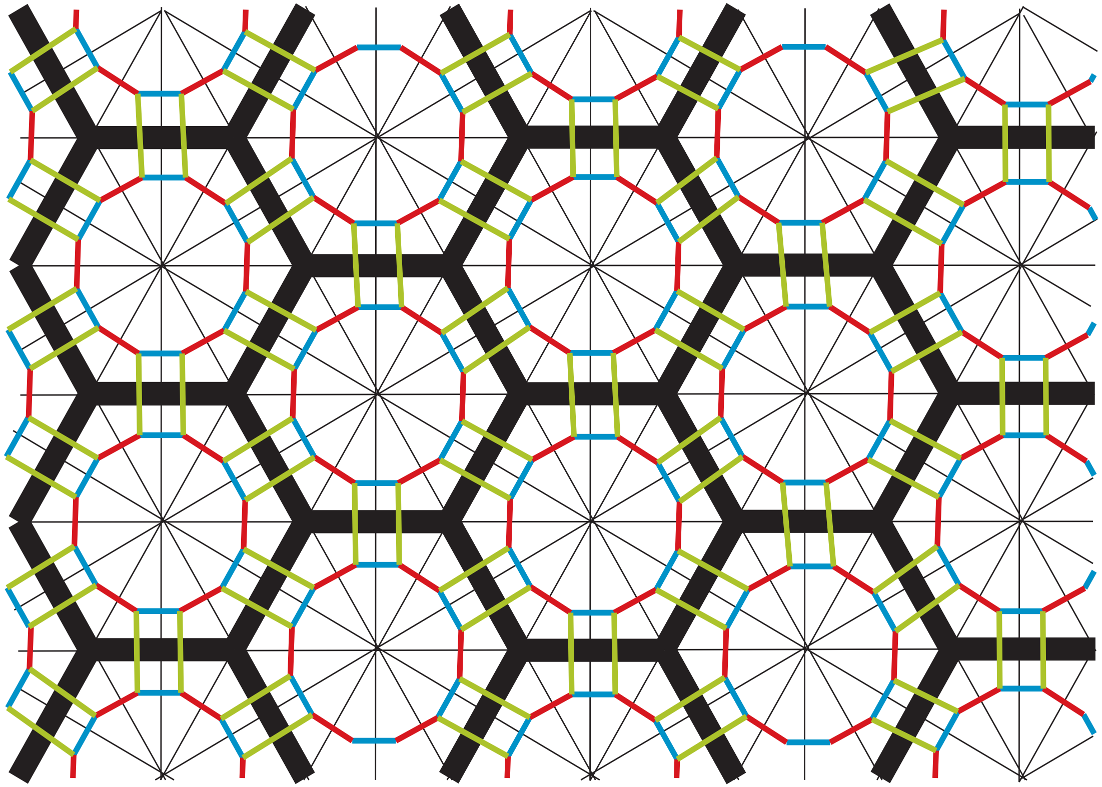

Delaney-Dress symbol

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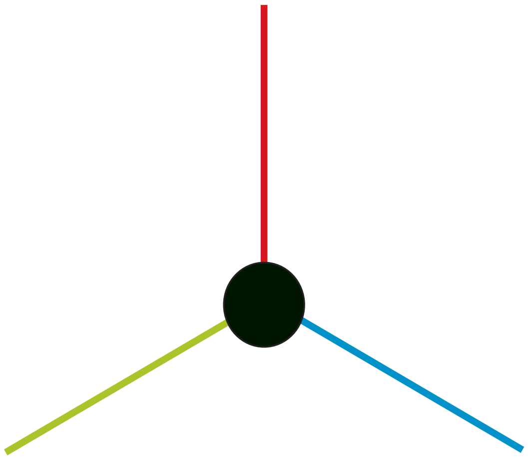

regular

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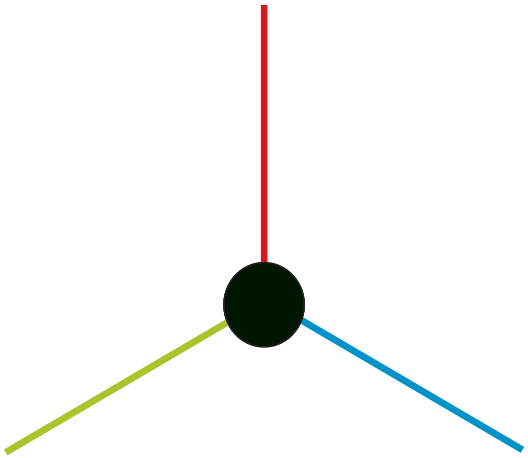

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chiral

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- On planar tessellations
Edge-homogeneous $\Longleftrightarrow$ edge-transitive

Spherical tessellations

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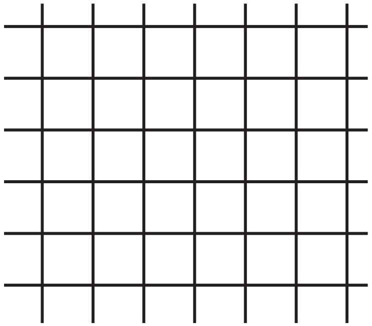
Euclidean tessellations



<4,4;4,4>

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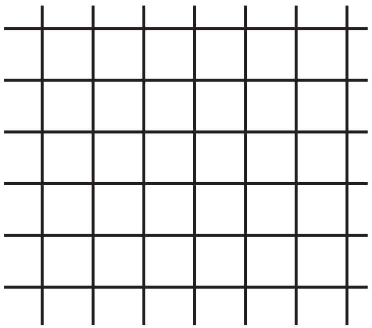

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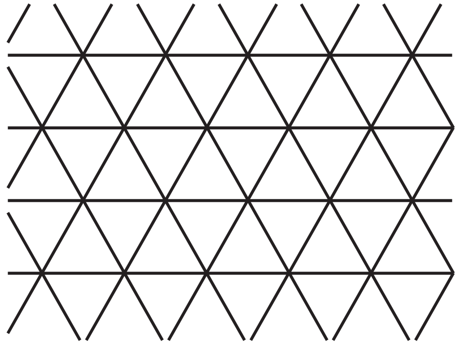

<3,3;6,6>

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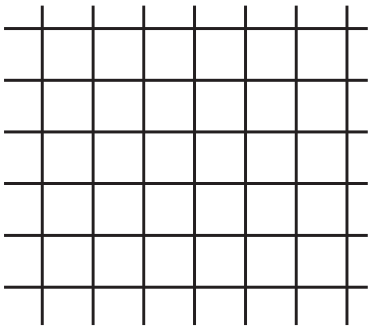

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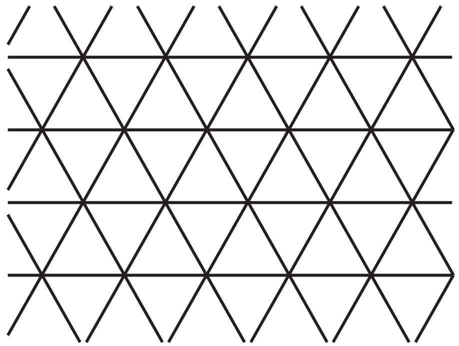

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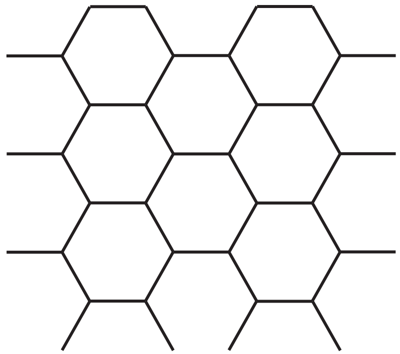

<6,6;3,3>

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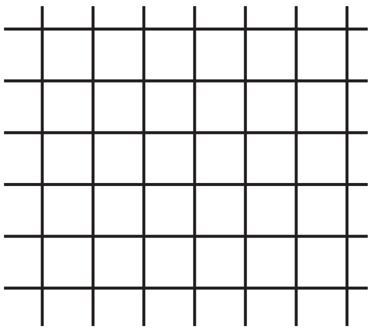
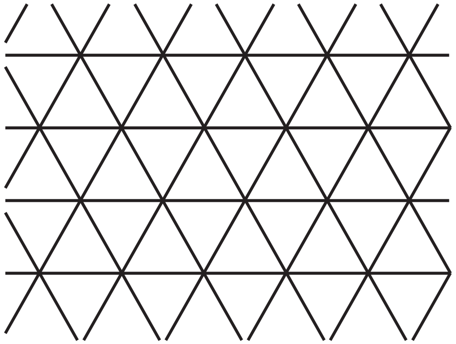


<4,4;4,4>



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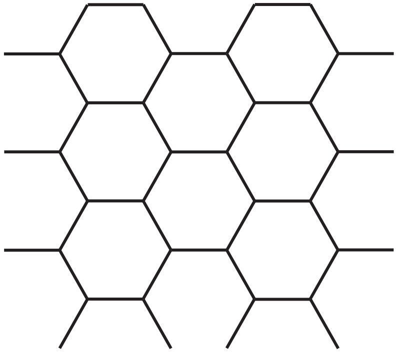

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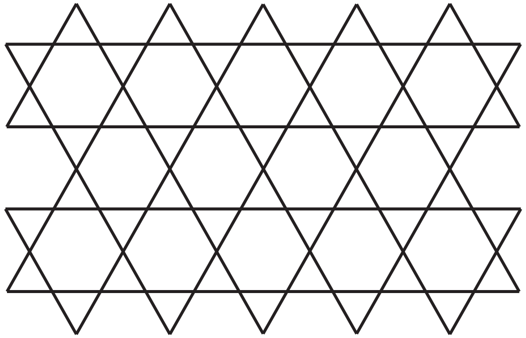

<3,6;4,4>

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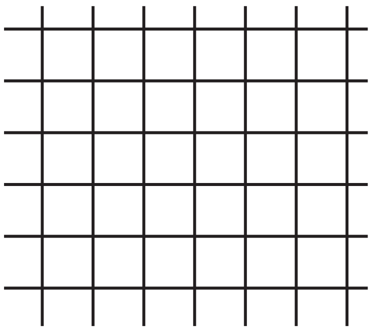

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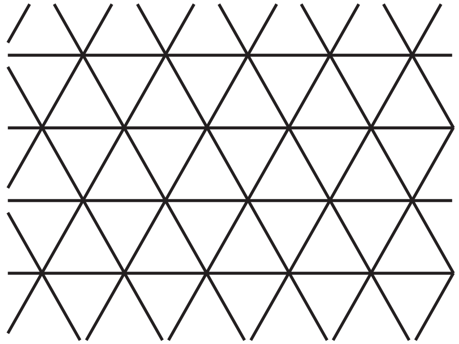

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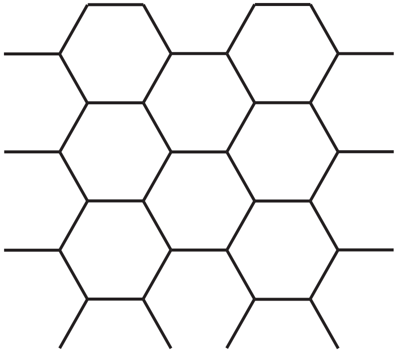

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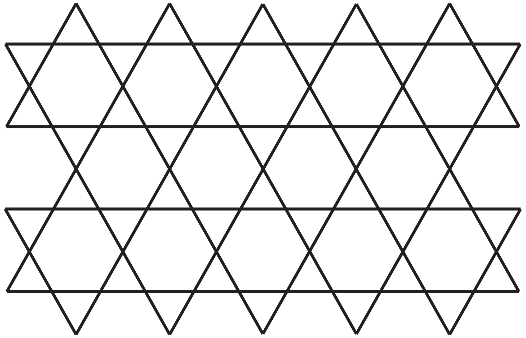

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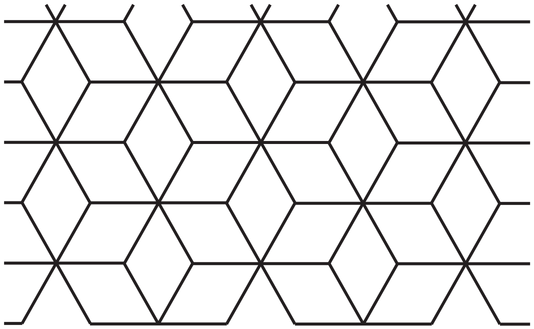
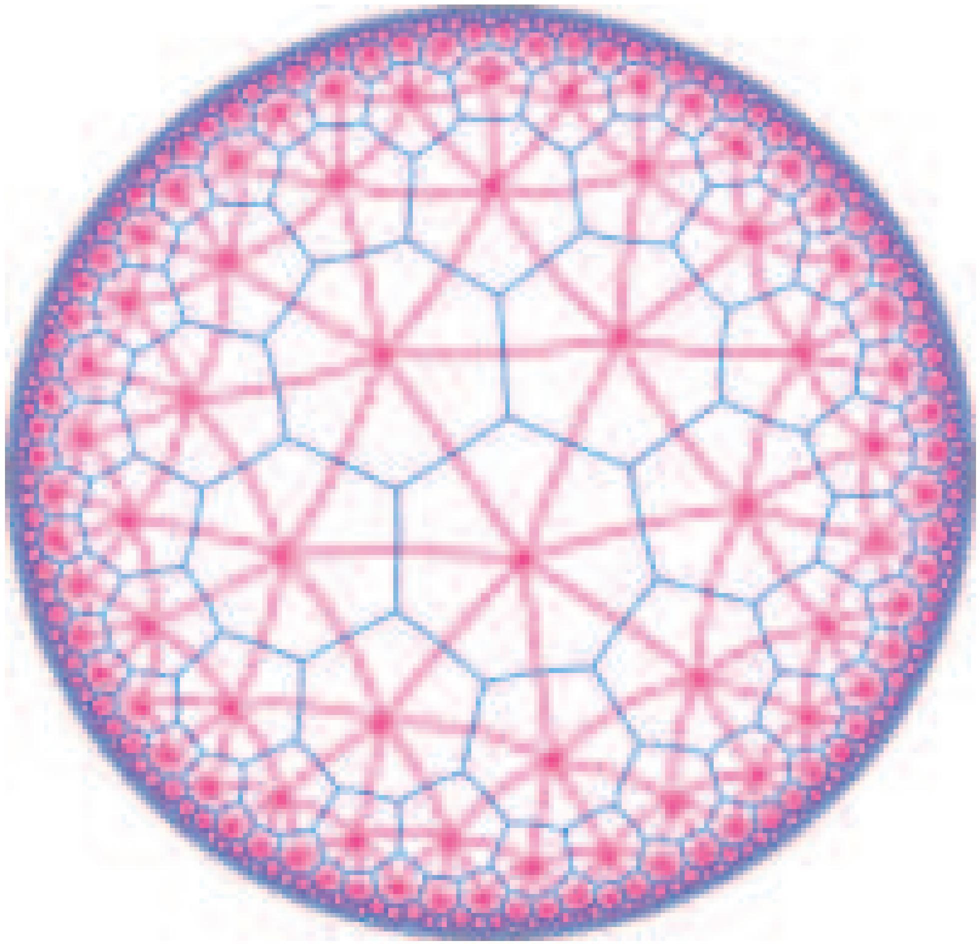


<4,4;3,6>

Hyperbolic tessellations



{3,7}

{7,3}

Admissible types

1 $p = q, s = t$: regular (reflexible)

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Compact surfaces

- Edge-transitive maps on compact surfaces

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- Edge-transitive maps on compact surfaces $\longrightarrow$ quotients of an edge-transitive planar tessellation

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Compact surfaces

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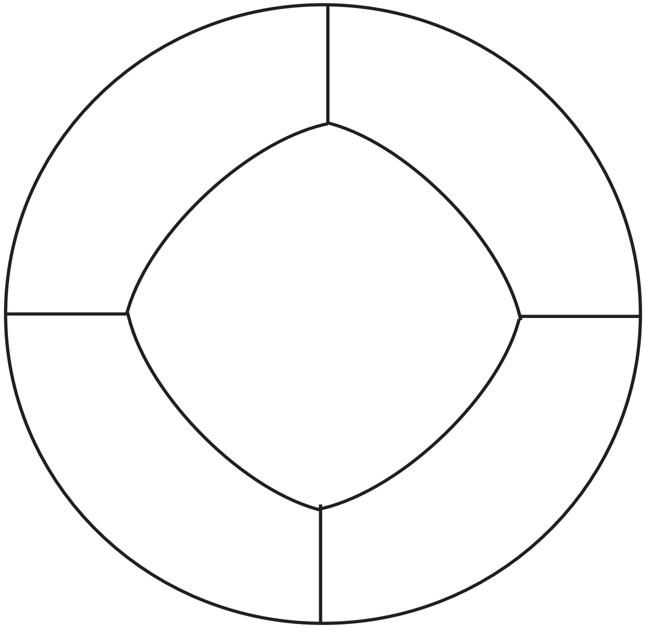
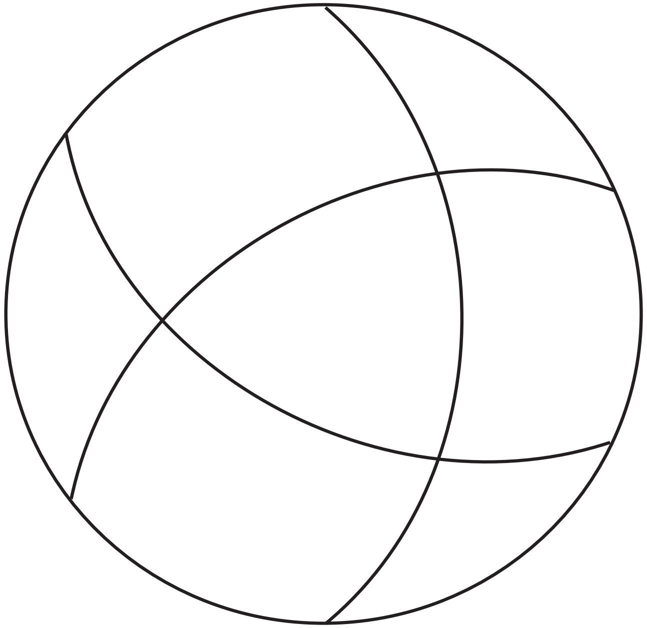
- Quotients of the sphere $\longrightarrow$ Projective plane
- Quotients of the Euclidean plane $\longrightarrow$ Torus, Klein bottle

Projective plane





Projective plane

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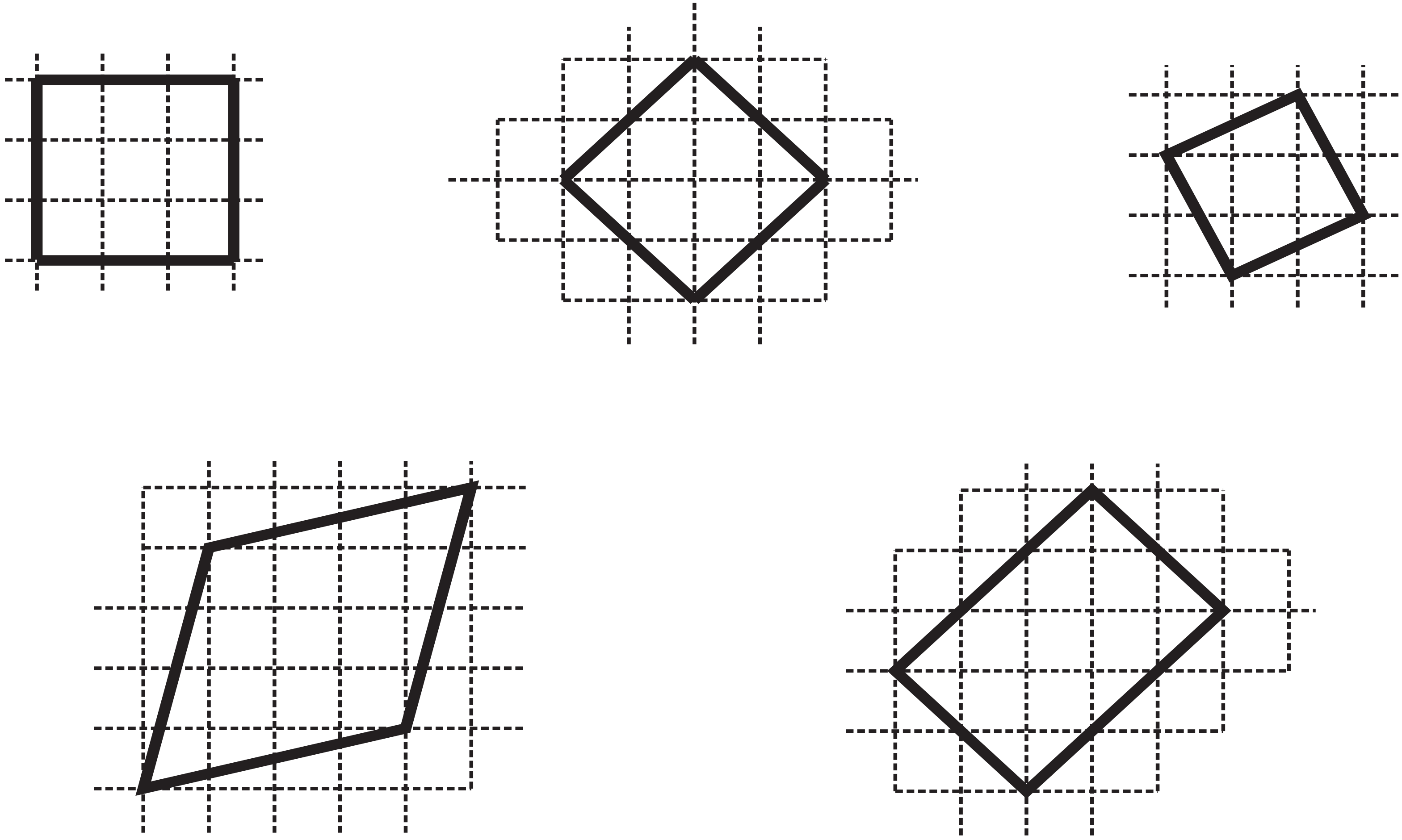
Torus



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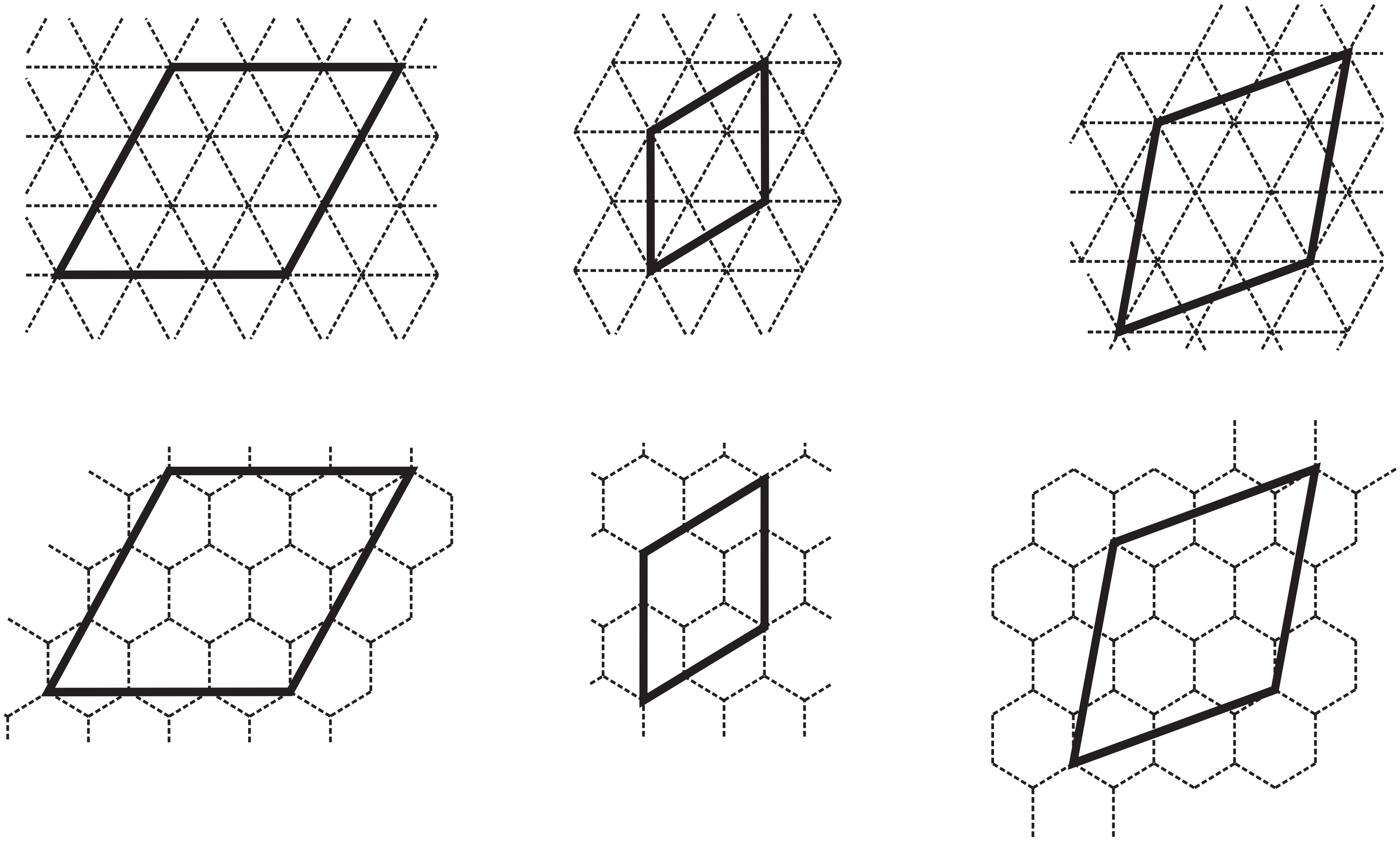

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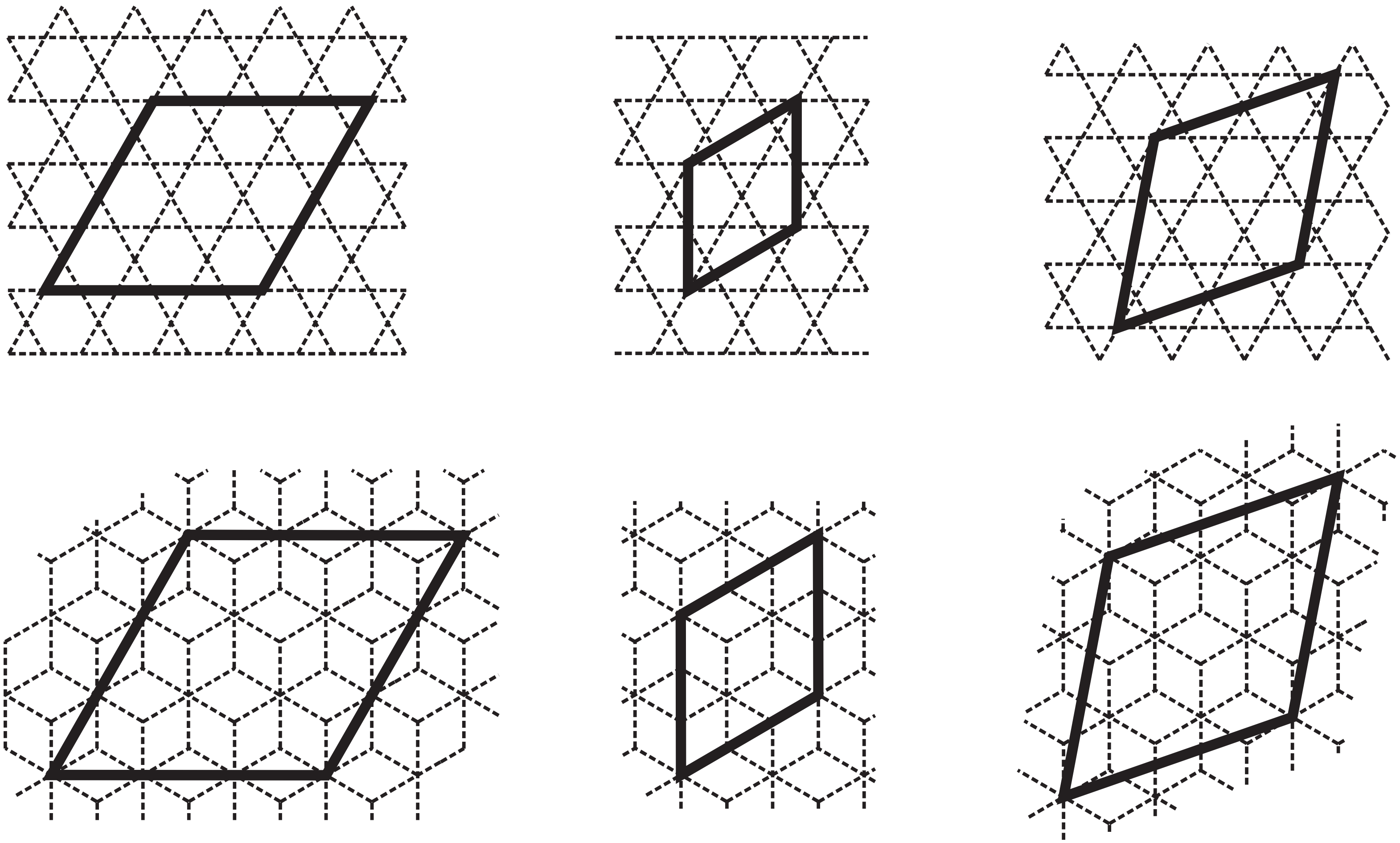
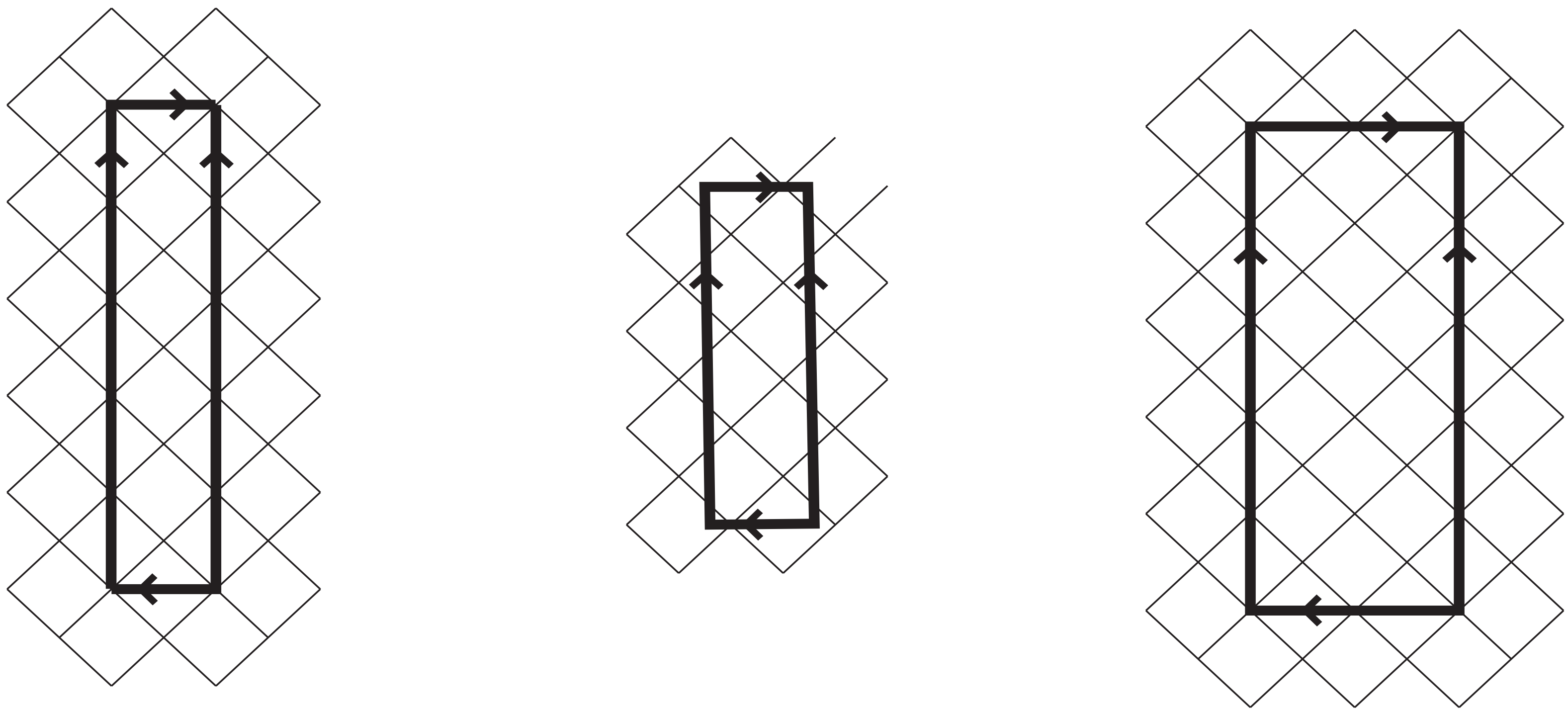


Klein bottle



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Admissible types

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- There are infinitely many maps on compact closed surfaces with any given hyperbolic type

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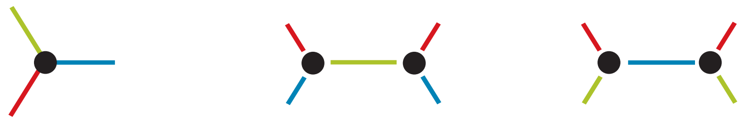

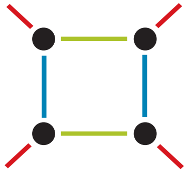

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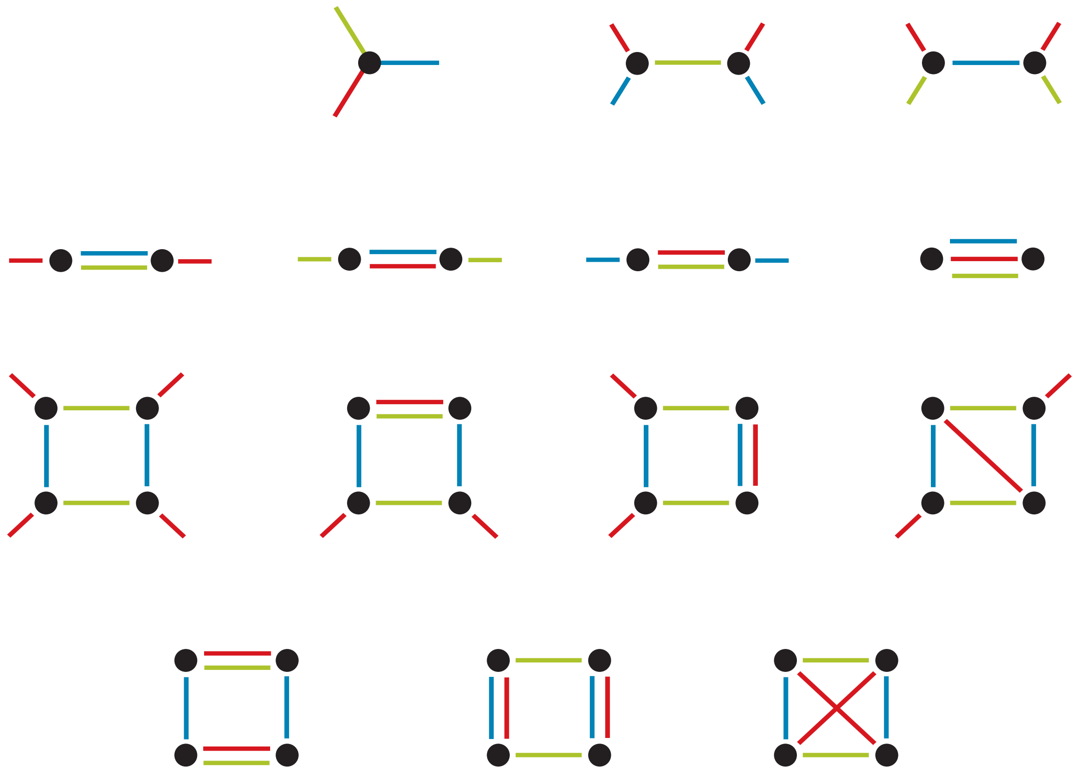

Higher genus

- Alen’s Orbanić’s PhD

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- Edge type

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  - Edge type
  - Delaney-Dress graph