THE 2-GROUPS OF ODD STRONG SYMMETRIC GENUS

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The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts faithfully.

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In this talk, we restrict our attention to 2-groups. The 2-groups are interesting in this context because of the well-known conjecture that, among the finite groups, almost all groups are 2-groups.

Important 2-groups

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For $m \geq 4$, let $QD(m)$ be the *quasidihedral* group with generators X, Y and defining relations

$$X^{2^{m-1}} = Y^2 = 1, YXY = X^{-1+2^{m-2}}. \tag{2}$$

Important 2-groups II

For $m \geq 4$, let $QA(m)$ be the *quasiabelian* group with generators X, Y and defining relations

$$X^{2^{m-1}} = Y^2 = 1, Y^{-1}XY = X^{1+2^{m-2}}. \tag{3}$$

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May & Zimmerman, 2000.
$\sigma^0(DC(m)) = 2^{m-2}$
$\sigma^0(QD(m)) = 2^{m-3}$
$\sigma^0(QA(m)) = 2^{m-2} - 1$

Even Genus

Theorem

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Next we consider a 2-group G acting on a Riemann surface of genus $g \equiv 3 \ (mod\ 4)$.

Theorems

Theorem

Let G be a group of order 2^m that acts on a Riemann surface W of genus $g \equiv 3 \pmod 4$ preserving orientation. Then G contains an element of order 2^{m-2} or larger. Further, if G is the quotient of the Fuchsian group Γ by the surface group K such that $W = U/K$, then in the signature of Γ, there are an odd number of ordinary periods that equal 2^{m-2}.

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Theorem

Let G be a group of order 2^m. If $\sigma^0(G) \equiv 3 \pmod 4$, then either $G \cong QA(m)$ or $Exp(G) = 2^{m-2}$.

Groups with cyclic subgroups of index 4

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Theorem

Burnside, 1911. *There are 14 families of groups G of order 2^m which contain a normal cyclic subgroup of index 4 in G.*

Partial Presentations

Ten of Burnside's 14 groups have abelian subgroup L of index 2. Then we must have $L \cong Z_2 \times Z_{2^{m-2}}$, and the group G has partial presentation

$$G = \langle\, x, y, z \mid x^{2^{m-2}} = y^2 = [x, y] = 1, \cdots \rangle. \qquad (4)$$

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In each of the remaining four of Burnside's groups, the subgroup L of index 2 is quasi-abelian with presentation

$$G = \langle\, x, y, z \mid x^{2^{m-2}} = y^2 = 1, y^{-1}xy = x^{1+2^{m-3}} \cdots \rangle. \tag{5}$$

List of Groups - Normal Case

Table: Non-abelian groups with cyclic subgroup of index 4

Family	Presentation	$z^{-1}xz =$	$z^{-1}yz =$	$z^2 =$
B_2	4	x^{-1}	y	y
B_4	4	$x^{-1+2^{m-3}}$	y	y
B_6	5	$x^{-1+2^{m-4}}$	y	y
B_7	4	x^{-1}	y	$x^{2^{m-3}}$
B_{11}	4	x^{-1}	$yx^{2^{m-3}}$	1
B_{12}	4	x^{-1}	$yx^{2^{m-3}}$	$yx^{2^{m-4}}$
B_{14}	5	$x^{-1+2^{m-3}}$	$yx^{2^{m-3}}$	1

Non-normal subgroups

Miller, 1901, 1902. Now assume that the non-abelian 2-group G has cyclic subgroups of index 4, but that none of these is normal. Again, we assume that G does not have a cyclic subgroup of index 2. If $m \geq 6$, there are exactly 11 non-isomorphic groups. These are enumerated in two articles of Miller .

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Six of Miller's groups have $L \cong Z_2 \times Z_{2^{m-2}}$. We denote these groups $M_i(m)$ for $1 \leq i \leq 6$. If $L = N(H)$ is quasi-abelian with presentation (5). There are also six of groups are of this type, which we denote $Q_i(m)$ for $1 \leq i \leq 6$. In addition, the groups $Q_3(m)$ and $M_4(m)$ are isomorphic.

List of Groups - Non-Normal Case

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Family	Presentation	$z^{-1}xz =$	$z^{-1}yz =$	$z^2 =$
M_3	4	$x^{-1+2^{m-4}}y$	$yx^{2^{m-3}}$	1
M_5	4	$x^{-1+2^{m-3}}y$	y	$x^{2^{m-3}}$
M_6	4	$x^{-1}y$	y	1
Q_2	5	$x^{-1}y$	$yx^{2^{m-3}}$	1
Q_5	5	$x^{-1+2^{m-4}}y$	y	1
Q_6	5	$x^{-1+2^{m-4}}y$	y	$x^{2^{m-3}}$

Theorems

Theorem

Let G be a group of order 2^m *with strong symmetric genus* $\sigma^0(G) \geq 2$. *Suppose* $Exp(G) = 2^{m-2}$ *and* $\Omega_{m-3} \neq G$. *If* $\sigma^0(G) < 1 + 3|G|/8$, *then* $\sigma^0(G) \equiv 1 \ (mod\ 4)$.

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Theorem

Let G be a 2-group with strong symmetric genus $\sigma^0(G) \geq 2$. *Suppose G has rank 3, and, further,* $Exp(G) = 2^{m-2}$ *and* $\Omega_{m-3} \neq G$. *If* $\sigma^0(G) < 1 + |G|/2$, *then* $\sigma^0(G) \equiv 1 \pmod 4$.

Groups of genus congruent to 3 modulo 4

Theorem

Let G be a group of order 2^m, with $m \geq 6$. Then $\sigma^0(G) \equiv 3 \pmod 4$ if and only if G is isomorphic to one of the following fourteen groups: $QA(m)$, $B_2(m)$, $B_4(m)$, $B_6(m)$, $B_7(m)$, $B_{11}(m)$, $B_{12}(m)$, $B_{14}(m)$, $M_3(m)$, $M_5(m)$, $M_6(m)$, $Q_2(m)$, $Q_5(m)$, $Q_6(m)$.

Genus Formulas

Strong Symmetric Genus of the Fourteen Families	
Family	Genus Formula
B_7	$2^{m-1} - 1$
QA, B_2, B_4, B_{11}, M_5	$2^{m-2} - 1$
M_6, Q_2	$2^{m-3} - 1$
B_{12}, B_{14}	$3 \cdot 2^{m-3} - 1$
M_3, Q_5	$3 \cdot 2^{m-4} - 1$
B_6, Q_6	$5 \cdot 2^{m-4} - 1$

Definitions

Let T be the set of integers $g \geq 2$ for which there is a 2-group of strong symmetric genus g. For an integer n, let $f(n)$ denote the number of integers in T that are less than or equal to n. Then the natural density $\delta(T)$ of T in the set of positive integers is

$$\delta(T) = \lim_{n\to\infty} \frac{f(n)}{n}.$$

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Let T_e and T_3 be the subsets of T consisting of the even integers and the integers congruent to 3 $(mod\ 4)$, respectively, with the companion "counting" functions denoted f_e and f_3, respectively.

Density Theorems

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Theorem

Almost all positive integers that are the genus of a 2-group are congruent to $1 \pmod 4$. *Further, the density* $\delta(T) \leq 1/4$.

Proof

Proof. Again assume $n = 2^m$, with $m \geq 4$. We need to count the groups in Theorem 7 that have genus n or less. Let G be one of these 2-groups. Then from the basic lower bound for the genus of a 2-group, we have

$$|G| \leq 16(\sigma^0(G) - 1) \leq 16(n-1) < 2^{m+4}, \qquad (6)$$

so that $|G| \leq 2^{m+3}$.

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Now, in each family, the only possible groups with genus in the range are those of order 16, 32, 64, ..., 2^{m+3}. Therefore, in each family, there are at most m groups with genus in the range. Since by Theorem 7, there are fourteen families, we have $f_3(n) = f_3(2^m) \leq 14m$, and thus $\delta(T_3) = 0$.

Definitions

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$$t(n) = \sum_{k=1}^{n} gnu(k)$$

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$$t(n) = \sum_{k=1}^{n} gnu(k)$$

Now let F_2 be the family of finite 2-groups, with companion counting function f_2. Define the density $\Delta(F_2)$ as

$$\Delta(F_2) = \lim_{n\to\infty} \frac{f_2(n)}{t(n)}.$$

The D2G Conjecture

Now $f_2(2047) = 49,497,918,395$ and $t(2047) = 49,910,536,613$, with the fraction $f_2(2047)/t(2047) \cong .992$. Most of the groups involved here are groups of order 1024, as $gnu(1024) = 49,487,365,422$. Anyway, this data suggests, perhaps, the following.

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Conjecture

D2G Conjecture. *The group density of the 2-groups is 1, that is,* $\Delta(F_2) = 1$.

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Let F_1 be the family of finite groups, each of which has strong symmetric genus congruent to 1 (mod 4), with companion counting function f_1.

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Theorem

If the D2G Conjecture holds, then the group density $\Delta(F_1) = 1$.