Extendability of group actions on non-orientable surfaces

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Abstract

Suppose the finite group G acts faithfully on some compact non-orientable surface S. Under what conditions does this action extend to a faithful action of some larger group on the same surface? This question will be considered, with particular attention to the case where the group G is cyclic. If such a cyclic group action is realised by means of a non-maximal NEC signature, then the action always extends, but in some other cases, the group G can be shown to be the full automorphism group of S. We can also find, for example, the largest cyclic group that is the full automorphism group of such a surface of given algebraic genus g, and the smallest algebraic genus of a non-orientable surface on which a given cyclic group C_n acts as the full automorphism group, or indeed the entire genus spectrum of such actions of C_n.

General question

Suppose G is a group of automorphisms of a surface S (which might be orientable or non-orientable, with our without boundary).

Question: How can we tell if G is the *full* automorphism group of S, or whether the action of G can be *extended* to the (faithful) action of some larger group L?

We answered this first for *cyclic group actions* on compact Riemann surfaces of genus > 1 in *J. London Math. Society* 59 (1999), 573–584, and then for *general groups acting on compact orientable surfaces* of genus > 1 in *Transactions Amer. Math. Society* 355 (2003), 1537–1557.

Actions on compact non-orientable surfaces

A compact non-orientable surface S of algebraic genus $g > 1$ can be considered as the quotient $\mathbb{H}/\Lambda$ of the hyperbolic plane $\mathbb{H}$ under the action of a fixed point-free proper non-Euclidean crystallographic group (NEC group) Λ.

Every group of automorphisms of S is then isomorphic to the quotient Γ/Λ for some proper NEC group Γ containing Λ as a normal subgroup, with $\Gamma^{+}\Lambda = \Gamma$ (for non-orientability).

If G is not the group Aut(S) of all automorphisms of S, then Γ is properly contained with finite index in some other NEC group Γ', which also normalises Λ. The converse holds as well, and accordingly, the given question is closely related to the finite-index extendability of NEC groups.

Summary formulation of question

When does a smooth epimorphism $\theta\colon \Gamma \to G$ with kernel Λ extend to a smooth homomorphism $\theta : \Gamma' \to G'$ (with the same kernel Λ) for some larger NEC group Γ'?

The extendability of any such action depends mainly on the signature, which encodes the geometry of a fundamental region for Γ.

In particular, although Γ could be contained in an NEC group Γ' normalising Λ, the group Γ might be abstractly isomorphic to a *maximal* NEC group — a group that is not contained as a subgroup of finite index in any other NEC group.

Non-Euclidean crystallographic groups

An NEC group is a co-compact discrete subgroup of the group of orientation-preserving or -reversing isometries of the hyperbolic plane $\mathbb{H}$.

Such a group Γ is generated by

- Elliptic elements x_i, for $1 \leq i \leq r$;
- Reflections $c_{i0}, \ldots, c_{is_i}$, for $1 \leq i \leq k$;
- Orientation-preserving elements e_i, for $1 \leq i \leq k$; and
- either Hyperbolic elements a_i, b_i, for $1 \leq i \leq \gamma$ $(+$ case)

  or Glide reflections d_i, for $1 \leq i \leq \gamma$ $(-$ case) subject to defining relations

$x_i^{m_i} = 1$ for $1 \leq i \leq r$;

${c_{ij-1}}^2 = {c_{ij}}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$ for $1 \leq j \leq s_i$, for $1 \leq i \leq k$;

$c_{is_i} = e_i c_{i0} e_i^{-1}$ for $1 \leq i \leq k$; and

$x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 {a_1}^{-1} {b_1}^{-1} \ldots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1$ in case $+$

or $x_1 \ldots x_r e_1 \ldots e_k {d_1}^2 \ldots {d_\gamma}^2 = 1$ in case $-$.

The orientation-preserving subgroup Γ^+ consists of all words (on the generators) in which the total number of occurrences of the reflections c_{ij} and glide reflections d_i is even.

The signature of Γ is then

$$\sigma(\Gamma) = \Big(\gamma;\ \pm;\ [m_1, ..., m_r];\ \{(n_{11}, ..., n_{1s_1}), ..., (n_{k1}, ..., n_{ks_1})\}\Big).$$

Riemann-Hurwitz formula

The area of a fundamental region for the NEC group Γ with given signature is $2\pi\mu(\Gamma)$, where

$$\mu(\Gamma) = \alpha\gamma + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right),$$

with $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise.

If Λ is a subgroup of finite index in Γ, then Λ is also an NEC group, and its area is given by $\mu(\Lambda) = |\Gamma : \Lambda| \cdot \mu(\Gamma)$. This is the Riemann-Hurwitz formula.

In particular, if $\Lambda \lhd \Gamma$ and $\Gamma/\Lambda = G$, then $\mu(\Lambda) = |G|\mu(\Gamma)$.

Non-maximal signatures

If every NEC group Γ with signature σ is properly contained in some other NEC group Γ', with finite index, and the dimensions of the Teichmüller spaces of Γ and Γ' coincide, then σ is called a non-maximal signature.

Singerman (1972) determined all non-maximal signatures for Fuchsian groups.

Bujalance (1982) determined all normal pairs (possibilities for the signatures of Γ and Γ' with $\Gamma \lhd \Gamma'$), and then later Estévez and Izquierdo (2006) found all non-normal pairs (viz. possibilities with $\Gamma \not\lhd \Gamma'$).

Key observation in the case where G is cyclic

If Γ is a proper NEC group admitting a smooth epimorphism $\theta\colon \Gamma \to G$ onto a cyclic group G, then the signature of Γ has no link periods — that is, $n_{ij} = 1$ for all i, j.

Proof. Let $G = C_n = \langle v \,|\, v^n = 1 \rangle$. If c_{j-1} and c_j are the canonical reflections associated with any link period n_{ij}, so that ${c_{j-1}}^2 = {c_j}^2 = (c_{j-1}c_j)^{n_{ij}} = 1$, then n must be even and $\theta(c_{j-1}) = \theta(c_j) = v^{n/2}$ (the unique element of order 2 in C_n). But then $\theta(c_{j-1}c_j) = v^{n/2}v^{n/2} = 1$, and so the smoothness of θ implies that $n_{ij} = 1$.

This reduces the number of pairs $(\sigma(\Gamma), \sigma(\Gamma'))$ to consider from the Bujalance and Estévez-Izquierdo lists to just 15.

Table of signature pairs

	Signature $\sigma = \sigma(\Gamma)$	Signature $\sigma' = \sigma(\Gamma')$	$\lvert\Gamma' : \Gamma\rvert$
Case 1	$(3; -; [-]; \{-\})$	$(0; +; [2,2,2]; \{(-)\})$	2
Case 2	$(2; -; [t]; \{-\})$	$(0; +; [2,2]; \{(t)\})$	2
Case 3	$(2; -; [-]; \{(-)\})$	$(0; +; [2,2]; \{(2,2)\})$	2
Case 4	$(1; +; [-]; \{(-)\})$	$(0; +; [2,2,2]; \{(-)\})$	2
Case 5	$(1; -; [t]; \{(-)\})$	$(0; +; [2]; \{(2,2,t)\})$	2
Case 6	$(1; -; [-]; \{(-),(-)\})$	$(0; +; [2]; \{(2,2,2,2)\})$	2
Case 7	$(1; -; [t,u]; \{-\})$, $\max(t,u) \geq 3$	$(0; +; [2]; \{(t,u)\})$	2
Case 8	$(1; -; [t,t]; \{-\})$, $t \geq 3$	$(0; +; [2,t]; \{(-)\})$	2
Case 9	$(0; +; [-]; \{(-),(-),(-)\})$	$(0; +; [-]; \{(2,2,2,2,2,2)\})$	2
Case 10	$(0; +; [t]; \{(-),(-)\})$	$(0; +; [-]; \{(2,2,2,2,t)\})$	2
Case 11	$(0; +; [t,u]; \{(-)\})$, $\max(t,u) \geq 3$	$(0; +; [-]; \{(2,2,t,u)\})$	2
Case 12	$(0; +; [t,t]; \{(-)\})$ $t \geq 3$	$(0; +; [t]; \{(2,2)\})$	2
Case 13	$(0; +; [t,t]; \{(-)\})$ $t \geq 3$	$(0; +; [t,2]; \{(-)\})$	2
Case 14	$(1; -; [t,t]; \{-\})$ $t \geq 3$	$(0; +; [-]; \{(2,2,2,t)\})$	4
Case 15	$(0; +; [t,t]; \{(-)\})$ $t \geq 3$	$(0; +; [-]; \{(2,2,2,t)\})$	4

Case-by-case analysis

For each of the 15 pairs $(\sigma(\Gamma), \sigma(\Gamma'))$ on the list, we look at how Γ sits inside Γ' in order to determine whether/when a smooth epimorphism $\theta\colon \Gamma \to C_n$ will extend to a smooth homomorphism $\theta : \Gamma' \to G'$ (for some G') with the same kernel. This is equivalent to the kernel Λ being normal in Γ' — a condition which can be checked in various ways.

In many cases, we find that $\theta\colon \Gamma \to C_n$ always extends to a smooth homomorphism $\theta\colon \Gamma' \to D_n$ (dihedral of order $2n$).

The other cases are more interesting.

Example: case 7

$\sigma(\Gamma) = (1; -; [t, u]; \{-\}), \quad \sigma(\Gamma') = (0; +; [2]; \{(t, u)\})$

Here the group Γ is generated by elements d, x and y such that $d^2xy = x^t = y^u = 1$, while Γ' is generated by involutions x_1, c_0 and c_1 such that $(c_0c_1)^t = (c_1x_1c_0x_1)^u = 1$.

An embedding of Γ into Γ' is given by $d \mapsto x_1c_0$, $x \mapsto c_0c_1$ and $y \mapsto c_1x_1c_0x_1$.

Conjugation by c_0 is an involutory automorphism of Γ with $d^{c_0} = d^{-1}$ and $x^{c_0} = x^{-1}$ and $y^{c_0} = xdy^{-1}x^{-1}d^{-1}$, which is simply inversion modulo $[\Gamma, \Gamma]$.

It follows that an extension of $\theta \colon \Gamma \to C_n$ is always possible, to a smooth epimorphism $\theta' \colon \Gamma' \to D_n$.

Example: case 8

$\sigma(\Gamma) = (1; -; [t,t]; \{-\})$, $\sigma(\Gamma') = (0; +; [2,t]; \{(-)\})$, $t \geq 3$

Here the group Γ is generated by elements d, x and y such that $d^2xy = x^t = y^t = 1$, while Γ' is generated by elements x_1, x_2 and c such that ${x_1}^2 = {x_2}^t = c^2 = [x_1x_2, c] = 1$.

There is a unique embedding of Γ as a subgroup of index 2 in Γ' given by $d \mapsto x_1c{x_2}^{-1}$, $x \mapsto x_2$ and $y \mapsto x_1x_2x_1$.

Conjugation by x_1 is an involutory automorphism of Γ such that $d^{x_1} = x^{-1}dx$ while $x^{x_1} = y$ and $y^{x_1} = x$. So $\theta\colon \Gamma \to C_n$ extends if and only C_n has an involutory automorphism that interchanges $\theta(x)$ and $\theta(y)$ while centralising $\theta(d)$.

This is possible in some cases (e.g. if $\theta(y) = \theta(x)^{-1}$ and $\theta(d) = 1$, or if $\theta(d) = \theta(x)^{n/2}$), but not in others.

Summary table of case-by-case analysis

	Signature $\sigma = \sigma(\Gamma)$	Signature $\sigma' = \sigma(\Gamma')$	Extends?
Case 1	$(3; -; [-]; \{-\})$	$(0; +; [2, 2, 2]; \{(-)\})$	Always
Case 2	$(2; -; [t]; \{-\})$	$(0; +; [2, 2]; \{(t)\})$	Always
Case 3	$(2; -; [-]; \{(-)\})$	$(0; +; [2, 2]; \{(2, 2)\})$	Always
Case 4	$(1; +; [-]; \{(-)\})$	$(0; +; [2, 2, 2]; \{(-)\})$	Always
Case 5	$(1; -; [t]; \{(-)\})$	$(0; +; [2]; \{(2, 2, t)\})$	Always
Case 6	$(1; -; [-]; \{(-), (-)\})$	$(0; +; [2]; \{(2, 2, 2, 2)\})$	Always
Case 7	$(1; -; [t, u]; \{-\})$, $\max(t, u) \geq 3$	$(0; +; [2]; \{(t, u)\})$	Always
Case 8	$(1; -; [t, t]; \{-\})$, $t \geq 3$	$(0; +; [2, t]; \{(-)\})$	Sometimes
Case 9	$(0; +; [-]; \{(-), (-), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$	Always
Case 10	$(0; +; [t]; \{(-), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, t)\})$	Always
Case 11	$(0; +; [t, u]; \{(-)\})$, $\max(t, u) \geq 3$	$(0; +; [-]; \{(2, 2, t, u)\})$	Always
Case 12	$(0; +; [t, t]; \{(-)\})$ $t \geq 3$	$(0; +; [t]; \{(2, 2)\})$	Sometimes
Case 13	$(0; +; [t, t]; \{(-)\})$ $t \geq 3$	$(0; +; [t, 2]; \{(-)\})$	Sometimes
Case 14	$(1; -; [t, t]; \{-\})$ $t \geq 3$	$(0; +; [-]; \{(2, 2, 2, t)\})$	Sometimes
Case 15	$(0; +; [t, t]; \{(-)\})$ $t \geq 3$	$(0; +; [-]; \{(2, 2, 2, t)\})$	Sometimes

Surprising(?) theorem [EB, JC & MC]

Let C_n be a cyclic group acting with non-maximal signature on a non-orientable surface S. Then the action of C_n always extends to the action of a larger group on S.

Proof. The only cases in which the expected extension does not always occur are cases 8 and 14 (where Γ has signature $(1; -; [t, t]; \{-\})$) and cases 12, 13 and 15 (where Γ has signature $(0; +; [t, t]; \{(-)\})$). But the actions in those cases extend in the way described in cases 7 and 11 (with $u = t$).

Moreover, the action of C_n always extends at least to an action of the dihedral group D_n on the same surface.

This contrasts with the analogous theorem for orientable surfaces, where cyclic actions do not always extend.

Largest orders of cyclic full automorphism groups

It has been known for some time that the maximum order of a cyclic group of automorphisms acting on a non-orientable surface of algebraic genus g is $2g+2$ when g is even, and $2g$ when g is odd [Wendy Hall (1978), also EB (1983)].

When the upper bound is attained, the signature of the corresponding NEC Γ is

- $(0;+;[2,g+1];\{(-)\})$ if g is even, or
- $(0;+;[2,2g];\{(-)\})$ or $(1;-;[2,2g];\{-\})$ if g is odd.

In all cases, the NEC group is non-maximal and the action extends, so the cyclic group is not the full automorphism group of the surface [EB, Gromadzki & Turbek (2001)].

Our results give this:

The largest integer n for which C_n is the full automorphism group of some non-orientable surface of algebraic genus g is

$$n(g) = \begin{cases} g+1 & \text{if } \; g \equiv 1 \bmod 4 \\ \;\; g & \text{if } \; g \text{ is even} \\ g-1 & \text{if } \; g \equiv 3 \bmod 4. \end{cases}$$

There are at least two maximal NEC group signatures giving such an action of C_n whenever g is odd or $g \in \{6, 12, 30\}$, and just one (viz. $(0;+;[2,2,g];\{(-)\})$) for all other even g.

It follows also that if n is larger than the value indicated above, then any action of C_n on a non-orientable surface of algebraic genus g extends to an action of D_n on that surface.

Moreover:

A surface S of algebraic genus $g \geq 2$ is q-hyperelliptic if it admits an involutory automorphism ϕ such that the quotient surface $S/\langle\phi\rangle$ has algebraic genus q.

If $q = 0$ then S is hyperelliptic, while 1-hyperelliptic surfaces are usually called elliptic-hyperelliptic.

Our work shows that if S is a compact non-orientable surface of algebraic genus $g \geq 2$ whose full automorphism group is cyclic of the largest possible order for its genus g, but $g \neq 3, 6, 7, 12, 30$, then

- S is hyperelliptic whenever $g \not\equiv 3$ mod 4, while
- S is elliptic-hyperelliptic (but not hyperelliptic) whenever $g \equiv 3$ mod 4.

The full cross-cap genus of a group

The symmetric cross-cap number of a finite group G is the minimum topological genus of any compact non-orientable surface S (with empty boundary) on which G acts effectively as a group of automorphisms [Tucker (1991), May (2001)].

The symmetric cross-cap number of C_n is known for all n (Bujalance, 1983). Again, when this bound is attained, C_n is not the full automorphism group of the surface.

We define a new parameter, the full cross-cap genus of a group G, as the minimum algebraic genus of a non-orientable surface S on which G is the full automorphism group.

Question: *What is the full cross-cap genus of C_n?*

The full cross-cap genus of cyclic groups

Let $n = {p_1}^{e_1}{p_2}^{e_2}\dots{p_s}^{e_s}$ be the prime-power decomposition of n, such that $p_1 < p_2 < \dots < p_s$.

Then the full cross-cap genus of C_n is

(a) $2p_1p_2p_3 - p_1p_2 - p_1p_3 - p_2p_3 + 1$
when $n = p_1p_2p_3$ with $3 < p_1 < p_2 < p_3 < \frac{p_1(p_2-1)}{p_2-p_1}$,

(b) $2n - \frac{2n}{p_1} - p_1 + 1$
when $s > 1$ and $e_1 = 1$ and n is not of the form in (a),

(c) $2n - \frac{2n}{p_1}$ otherwise.

[The proof uses appropriately chosen maximal NEC groups.]

One more theorem [EB, JC, MC]

For each $n > 1$, there exists some g_0 such that C_n is the full automorphism group of some non-orientable surface of algebraic genus g for every $g \geq g_0$.

To prove this, for n even we take a maximal NEC group with signature $(\gamma; -; [n, \overset{r}{\ldots}, n]; \{(-)\})$ where $\gamma > 0$ and $\gamma + r > 2$, while for n odd we take a maximal NEC group with signature $(\gamma; -; [n, \overset{r}{\ldots}, n]; \{-\})$ where $\gamma > 0$ and $\gamma + r > 3$.

The algebraic genus of this surface is given by

$$g = 1 + |G|\mu(\Gamma) = \begin{cases} n(\gamma-1) + r(n-1) + 1 & \text{for } n \text{ even} \\ n(\gamma-2) + r(n-1) + 1 & \text{for } n \text{ odd.} \end{cases}$$

By varying γ and r, we can make g equal to any integer greater than $(n-1)^2$.

Thank You!