

Which $(2, 3, r)$ triangle groups give rise to the strong symmetric genus of some finite group?

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Strong Symmetric Genus

Definition

Given a finite group G , the smallest genus of any closed orientable topological surface on which G acts faithfully as a group of orientation preserving symmetries is called the **strong symmetric genus** of G .

- The strong symmetric genus of the group G is denoted $\sigma^0(G)$.
- By a result of Hurwitz [1893], if $\sigma^0(G) > 1$ for a finite group G , then $\sigma^0(G) \geq 1 + \frac{|G|}{84}$.

Generators and the Riemann-Hurwitz Equation

- If a finite group G has generators x and y of orders p and q respectively with xy having the order r , then we say that (x, y) is a (p, q, r) generating pair of G .
- The existence of a (p, q, r) generating pair gives a faithful orientation preserving action of the group G on a surface S .
- The genus of the surface S is found from the Riemann-Hurwitz formula:

$$\text{genus}(S) = 1 + \frac{|G|}{2} \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} \right).$$

Minimal Generating Pairs

- A (p, q, r) generating pair of G is called a minimal generating pair if no generating pair for the group G gives an action on a surface of smaller genus.
- The groups of small strong symmetric genus are well known.
- So generally we will assume that $\sigma^0(G) \geq 2$ or equivalently that any generating pair will be a (p, q, r) generating pair with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.
- With these assumptions a (p, q, r) generating pair is minimal if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ is the largest of any generating pair.

A Lemma by Singerman

Lemma

Let G be a finite group such that $\sigma^0(G) > 1$. If $|G| > 12(\sigma^0(G) - 1)$, then G has a (p, q, r) generating pair with

$$\sigma^0(G) = 1 + \frac{1}{2}|G| \cdot \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right).$$

- Singerman's Lemma implies that if G has a minimal (p, q, r) generating pair such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{5}{6}$, then the strong symmetric genus is given by this generating pair.
- Since $\sigma^0(G) > 1$, we know that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

More on Singerman's Lemma

- Recall: if G has a minimal (p, q, r) generating pair such that $\frac{5}{6} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then the strong symmetric genus is given by this generating pair.
- What triples of numbers (p, q, r) fit this requirement:
 - $(2, 3, r)$ for any r .
 - $(2, 4, r)$ for $5 \leq r \leq 11$.
 - $(3, 3, r)$ for $r = 4$ or $r = 5$.
- So all but finitely many such triples are of the form $(2, 3, r)$.
- The question we ask is which of these triples $(2, 3, r)$ have a group G with a minimal $(2, 3, r)$ generating pair.

Known Results

- May and Zimmerman (2003) have shown that there is a group of every strong symmetric genus.
 - They used the groups $\mathbb{Z}_k \times D_n$.
 - These groups do not have minimal $(2, 3, r)$ generating pair.
- Glover and Sjerne (1985) showed that the group $PSL_2(p)$ for the prime p has a minimal $(2, 3, d)$ generating pair
 - if $p \geq 13$, $p \not\equiv \pm 1 \pmod{5}$, and $p \not\equiv \pm 1 \pmod{8}$,
 - where d is the minimal integer at least 7 such that $d \mid \frac{p-1}{2}$ or $d \mid \frac{p+1}{2}$.
- For any prime $r \geq 7$ there is a prime p such that $PSL_2(p)$ has a minimal $(2, 3, r)$ generating pair.
 - Uses the results of Glover and Sjerne
 - and Dirichlet's theorem on primes in arithmetic progressions.

Generalized Symmetric Groups

- $G(n, m) = \mathbb{Z}_m \wr \Sigma_n$ for $n > 1$ and $m \geq 1$.
- $G(n, m)$ is the group of $n \times n$ matrices containing
 - the permutation matrices and
 - the diagonal matrices with entries in a multiplicative cyclic group of size m .
- $G(n, 1)$ is the traditional symmetric group Σ_n .
- $G(n, 2)$ is the hyperoctahedral group B_n .
- The strong symmetric genus has been found for the groups:
 - $G(n, 1)$ [Conder, 1980]
 - $G(n, 2)$ and $G(n, 3)$ [J, 2004 and 2008]
 - $G(3, m)$, $G(4, m)$ and $G(5, m)$ [Ginter, Johnson, McNamara, 2008]

D -type Generalized Symmetric Groups

- $D(n, m) = (\mathbb{Z}_m)^{n-1} \rtimes_{\theta} \Sigma_n$ for $n > 1$ and $m \geq 2$.
- $D(n, m) \triangleleft G(n, m)$ with an index of m .
- $D(n, m)$ is the group of $n \times n$ matrices containing
 - the permutation matrices and
 - the diagonal matrices of determinant 1 with entries in a multiplicative cyclic group of size m .
- $D(n, 2)$ is the Coxeter group D_n .
- The strong symmetric genus has been found for the groups $D(n, 2)$ [J, 2007]

AD-type Generalized Symmetric Groups

- $AD(n, m) = (\mathbb{Z}_m)^{n-1} \rtimes_{\theta} A_n$ for $n > 1$ and $m \geq 2$.
- $AD(n, m) \triangleleft D(n, m)$ with an index of 2.
- $AD(n, m) \triangleleft G(n, m)$ with an index of $2m$.
- $AD(n, m)$ is the group of $n \times n$ matrices containing
 - the permutation matrices of determinant 1 and
 - the diagonal matrices of determinant 1 with entries in a multiplicative cyclic group of size m .

Subgroups of Generalized Symmetric Group

- The strong symmetric genus of both D -type generalized symmetric groups and both AD -type generalized symmetric groups when n is 3, 4 or 5 have been found [Bowser, Partridge, Rodgers; 2009].
- The following results are useful in this talk:
 - When $3 \nmid m$, $D(3, m)$ has a minimal $(2, 3, 2m)$ generating pair.
 - When m is odd, $AD(4, m)$ has a minimal $(2, 3, 3m)$ generating pair.

Using the groups $D(3, m)$ and $AD(4, m)$

- We will generalize these results to include m either 0 or 1.
 - $D(3, 1) = \Sigma_3$ which has a minimal $(2, 2, 3)$ generating pair.
 - $AD(4, 1) = A_4$ which has a $(2, 3, 3)$ generating pair
 - $D(3, 0)$ and $AD(4, 0)$ are the trivial group.
- For any m and p such that $3 \nmid m$, p is odd, and $2m$ is relatively prime to $3p$, then $D(3, m) \times AD(4, p)$ has a minimal $(2, 3, 6m \cdot p)$ generating pair.

r divisible by 2 or 3

- Consider r which is at least 8 and is a multiple of 2 and/or 3 we can make a group with a $(2, 3, r)$ minimal generating pair as follows:
 - If $3 \nmid r$, then r is even and $D(3, \frac{r}{2})$ has a minimal $(2, 3, r)$ generating pair.
 - If $3|r$, write r as $3 \cdot k \cdot 2^h$ where k is odd:
 - If $h = 0$ then $AD(4, k)$ has a minimal $(2, 3, r)$ generating pair.
 - If $h > 0$ then $D(3, 2^{h-1}) \times AD(4, k)$ has a minimal $(2, 3, r)$ generating pair.

Prime r

- Recall from work of Glover and Sjerve we saw that if $r \geq 7$ is prime there exists a group $PSL_2(p)$ with a minimal $(2, 3, r)$ generating pair
- We find a prime p such that:
 - $p \geq 13$,
 - $p \not\equiv \pm 1 \pmod{8}$,
 - $p \not\equiv \pm 1 \pmod{9}$,
 - $p \not\equiv \pm 1 \pmod{q}$ for each prime q with $5 \leq q < r$, and
 - $p \equiv \pm 1 \pmod{r}$
- Such a prime exists by Dirichlet's theorem on primes in arithmetic progressions.
- This prime p gives a $PSL_2(p)$ with a minimal $(2, 3, r)$ generating pair.

$r = 5$ and $r = 25$

- Recall that A_5 has a minimal $(2, 3, 5)$ generating pair.
- In addition A_5 does not have any generating pairs of type $(2, 3, r)$ for $r > 5$.
- We also have a group with a minimal $(2, 3, 25)$ generating pair and no generating pairs of type $(2, 3, r)$ for $r > 25$.
 - There is a subgroup H of A_5 isomorphic to A_5 that is transitive.
 - The desired group is $G = (\mathbb{Z}_5)^5 \rtimes_{\theta} H$.
 - G has order 187,500.

Square Free r

- For any prime p such that
 - $p \geq 13$,
 - $p \not\equiv \pm 1 \pmod{5}$, and
 - $p \not\equiv \pm 1 \pmod{8}$.
- let d be the minimal integer at least 7 such that $d \mid \frac{p-1}{2}$ or $d \mid \frac{p+1}{2}$.
- The group $A_5 \times PSL_2(p)$ has a minimal $(2, 3, 5d)$ generating pair.
- If G is the group from the semidirect product from the last slide, the group $G \times PSL_2(p)$ has a minimal $(2, 3, 25d)$ generating pair.

Square Free r

- If r is not divisible by 2 or 3, r is square free or r is a multiple of 5 and $\frac{r}{5}$ is square free, then we can write r as a product of distinct primes q_i for $i = 1, 2, \dots, n$ each at least 7 and either 1, 5 or 25.
- for each q_i we find a prime p_i such that
 - $p_i \geq 13$,
 - $p_i \not\equiv \pm 1 \pmod{8}$,
 - $p_i \not\equiv \pm 1 \pmod{9}$, and
 - $p_i \not\equiv \pm 1 \pmod{s}$ for each prime $s \neq q_i$ such that $5 \leq s < r$.
- Let G be the trivial group, A_5 or the group of order 187,500 discussed earlier for the factor of 1, 5 or 25, respectively.
- Then $G \times \prod_{i=1}^n PSL_2(p_i)$ has a minimal $(2, 3, r)$ generating pair.

Summary

Theorem

Given an integer $r \geq 7$, there is a finite group G which has a minimal $(2, 3, r)$ generating pair if:

- *r is divisible by two or three,*
- *r is square free, or*
- *r is a multiple of 25 with $\frac{r}{5}$ square free.*

In each case above, the group G has a strong symmetric genus given by the Riemann-Hurwitz Equation for the $(2, 3, r)$ generating pair.

Small r Divisible by a Square

- Another result of Glover and Sjerne (1987) gives the minimal generating pair for the groups $PSL_2(q)$ where q is a prime power.
- Using this result I have search using Magma for a $PSL_2(q)$ with a minimal $(2, 3, r)$ generating pair for each of the remaining r that is less than 2695.