Pairs of symmetries of Riemann surfaces

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The main reason

X – compact Riemann surface;
Symmetry of X – an antiholomorphic involution in $G = \mathrm{Aut}^{\pm}(X)$.

$$\boxed{\text{Riemann surfaces } X} \longleftrightarrow \boxed{\begin{array}{l}\text{smooth complex projective}\\ \text{algebraic curves } \mathcal{C}_X\end{array}}$$

Moreover:

- Riemann surface X admits a symmetry σ if and only if the corresponding curve $\mathcal{C}_X$ has a real form $\mathcal{C}_X(\sigma)$;
- $\sigma \overset{\mathrm{Aut}^{\pm}(X)}{\sim} \tau \Leftrightarrow \mathcal{C}_X(\sigma) \overset{\mathbb{R}}{\cong} \mathcal{C}_X(\tau)$;
- The set $\mathrm{Fix}(\sigma)$ is homeomorphic to a smooth projective model of the corresponding real form $\mathcal{C}_X(\sigma)$.

NEC groups and Fuchsian groups

NEC group Λ – *discrete and cocompact* subgroup of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. Its algebraic presentation is determined by the *signature*:

$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k\}),$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$.

Fuchsian group Γ – an NEC group having no orientation reversing elements, hence it has signature of the abbreviated form

$$s(\Gamma) = (g; m_1, \ldots, m_r).$$

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$x_i, \ i = 1, \ldots, r$	$x_i^{m_i} = 1, \ i = 1, \ldots, r$
$c_{ij}, \ i = 1, \ldots, k$ $j = 0, \ldots, s_i$	$c_{ij}^2 = 1, \quad i = 1, \ldots, k$ $j = 0, \ldots, s_i$
	$(c_{ij-1}c_{ij})^{n_{ij}} = 1, i = 1, \ldots, k$ $j = 1, \ldots, s_i$

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$e_i, \; i = 1, \dots, k$	$c_{is_i} = e_i^{-1} c_{i0} e_i, \; i = 1, \dots, k$
$a_i, \; b_i, \; i = 1, \dots, h$	$x_1 \dots x_r e_1 \dots e_k [a_1, b_1] \dots [a_h, b_h] = 1$

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$e_i, \; i = 1, \ldots, k$	$c_{is_i} = e_i^{-1} c_{i0} e_i, \; i = 1, \ldots, k$
$a_i, \; b_i, \; i = 1, \ldots, h$	$x_1 \ldots x_r e_1 \ldots e_k [a_1, b_1] \ldots [a_h, b_h] = 1$
$d_i, \; i = 1, \ldots, h$	$x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 = 1.$

Hyperbolic area of the fundamental region of Λ is given by the formula:

$$\mu(\Lambda) = 2\pi(\varepsilon h + k - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) +$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - \frac{1}{n_{ij}})).$$

with $\varepsilon = 2$ or 1 according to the sign being $+$ or $-$.

[Hurwitz-Riemann formula] For a subgroup Λ' of finite index in an NEC group Λ we have

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda).$$

Fuchsian group Γ is called *surface* group if it is torsion free and in this case

$$s(\Gamma) = (g; -).$$

$X = \mathcal{H}/\Gamma$ is a Riemann surface of genus $g \geq 2$ and topology of X corresponds with the algebraic structure of the Fuchsian surface group Γ.

$G = \mathrm{Aut}^{\pm}(X) = \Lambda/\Gamma$ for some NEC group Λ and Fuchsian surface group Γ.

There is the *canonical* epimorphism $\theta : \Lambda \to G = \Lambda/\Gamma$

For σ being a symmetry of a Riemann surface X, the connected component of $\mathrm{Fix}(\sigma)$ shall be called an *oval* of σ.

[Harnack] The symmetry of a Riemann surface X of genus g has at most $g+1$ ovals.

We call σ a $(M-q)$-symmetry if σ has $g+1-q$ ovals. For $q=g+1$ we obtain a fixed point free symmetry.

[Gromadzki 1997] The number of ovals of a symmetry σ of X equals

$$\sum [C(G,\theta(c_i)) : \theta(C(\Lambda, c_i))]$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections c_i whose images under θ are conjugate to σ, where $\theta : \Lambda \to G$ denotes the canonical epimorphism.

σ, τ – two symmetries of X;
N – the order of $\sigma\tau$;
t – the total number of ovals of σ and τ;
$G = \mathrm{Aut}^{\pm}(X) = \mathrm{D}_N$.

[Bujalance, Costa, Singerman, Natanzon 1993] Two symmetries σ and τ have at most

$$\frac{4g}{N} + 2$$

ovals if N is even, and

$$\frac{2(g-1)}{N} + 4$$

ovals if N is odd. These bounds are attained for $N|4g$ or $N|g-1$ respectively.

[2006] Two symmetries σ and τ have at most

$$\left[\frac{4g}{N}\right] + 2$$

ovals if N is even, and

$$\left[\frac{2(g-1)}{N}\right] + 3$$

ovals if N is odd. These bounds are attained for infinitely many values of g.

Let σ, τ be two $(M-q)$- and $(M-q')$-symmetries of a Riemann surface of genus g with the product of order 2^n.

We define $\nu_g(q, q') =$ the maximal possible power of 2 that may be realized as the order of the product $\sigma\tau$.

For $g \geq 2, q$ and q' such that $q \leq q' \leq g$ the following conditions hold:

1. $\nu_g(q, q') = 2$ for $g \geq q + q' + 1$;
2. $\nu_g(q, q') \geq 4$ for $g \leq q + q'$ and $\{q, q'\} \neq \{1, g\}$ with $g > 2$;
3. $\nu_g(1, g) = 2$ for $g > 2$.

Assume now that $q \leq q' \leq g$, i.e. symmetries have ovals and $n > 2$. Then $t = 2g + 2 - q - q' \leq g/2^{n-2} + 2$.

[2009] If g, q, q' and n satisfy

$$2g - g/2^{n-2} \leq q + q' < 2g - g/2^{n-1} \tag{1}$$

then

$$\nu_g(q, q') \leq 2^n.$$

[2009] Let g, q, q' and n be integers satisfying conditions $2^{n-2}|g$ and $q + q' \geq 2g - g/2^{n-2}$. Then

$$\nu_g(q, q') \geq 2^n$$

and the equality holds for $q + q' < 2g - g/2^{n-1}$.

What happens if the lower bound on $q+q'$ in (1) is not an integer?

[2009] Let g, q, q' and $n \geq 3$ be integers such that $g = 2^{n-3}a$, a is odd and $q+q' \geq 2g - g/2^{n-2}$. Then

$$\nu_g(q,q') \geq 2^n$$

and the equality holds for $q+q' < 2g - g/2^{n-1}$ except for the case of $g=7, q=5, q'=7, n=3$ for which $\nu_7(5,7)=4$.

[2009] For $\{g,q,q'\} \neq \{7,5,7\}$ such that $q+q' \geq 3g/2$ we have $\nu_g(q,q') \geq 8$.

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[2009] For $\{g, q, q'\} \neq \{7, 5, 7\}$ such that $q + q' \geq 3g/2$ we have $\nu_g(q, q') \geq 8$.

It gets worse for smaller powers of 2:

[2009] Let $g = 2^k a$ for $1 \leq k < n - 3$ and odd a such that $2^{n-k-2} | a - 1$. Then for $q + q' = 2g - [g/2^{n-2}]$, $\nu_g(q, q') < 2^n$.

Now we shall consider the case when g is of the form $g = 2^{n-1}b + 1$ for some (possibly even) integer b and $n \geq 3$.

[2009] If $2g - 2b - 1/2^{n-2} \leq q + q' < 2g - b - 1/2^{n-1}$, then

$$2^{n-1} \leq \nu_g(q, q') \leq 2^n.$$

[2009] Let g, q and q' integers holding conditions given above. Then $\nu_g(q, q') = 2^n$ if and only if one of the following holds:
1: $q + q' \geq 2g + 2 - 2b$;
2: $q + q' = 2g + 1 - 2b$ and $q' \geq g - 1$ or $n \leq 4$;
3: $q + q' = 2g - 2b$ and $q' = g - 1$ or $n = 3$.

For $q = q' = g$ there is no n such that

$$2g - g/2^{n-2} \leq q + q' < 2g - g/2^{n-1}.$$

[2009] Let g and $n \geq 2$ hold $2^{n-1} \leq g < 2^n$. Then

$$\nu_g(g,g) \leq 2^n$$

and this bound is attained if and only if $g = 2^n - 2^{n-l}$ or $g = 2^n + 1 - 2^{n-l}$ for $0 < l < n$.

[2009] If $g = 2^k a$ or $g = 2^k a + 1$ for some $k \geq 1$ and odd a then

$$\nu_g(g,g) \geq 2^{k+1}$$

and the equality holds for $a = 1$.

Symmetries without ovals (or fpf symmetries)

Let X be a Riemann surface of genus g, having a pair of symmetries σ, τ, where τ is a fpf symmetry.

Let t denote the number of ovals of σ and N be the order of $\sigma\tau$

[Izquierdo, Singerman 1998]

1. If N is divisible by 4, then g is odd.

2. If g is even and $t > 0$, then N is even but not divisible by 4 and t is odd.

3. For any odd g and $t \leq g$ there exists Riemann surface of genus g, having a pair of non-commuting symmetries σ, τ.

[2008] Let X be a Riemann surface admitting a noncommuting pair σ, τ. Then:

(1) If $N = 4$, then $t \leq g$ and this bound is attained for arbitrary odd g;

(2) If $N \equiv 0 \mod 4$ and $N > 4$, then

$$t \leq \left[\frac{4(g-1)}{N}\right];$$

(3) If $N \equiv 2 \mod 4$, then

$$t \leq \left[\frac{2(g+1)}{N}\right] + 2.$$

Bounds in (2) and (3) are attained for infinitely many values of g.

Let X be a Riemann surface of odd genus g, having pair of noncommuting $(M-q)$- and fpf symmetries σ, τ with the product of order 2^n.

[2009] The following conditions hold:
(1) If $n > 2$ then

$$q \geq g + 1 - (g-1)/2^{n-2}$$

and the bound is attained when $(g-1)/2^{n-2}$ is even or $n = 3$;
(2) If $n = 2$, then

$$q \geq 1$$

and the bound is attained for any odd g.

[2009] If $g, q, n > 2$ hold

$$g + 1 - (g-1)/2^{n-2} \leq q < g + 1 - (g-1)/2^{n-1}, \qquad (2)$$

then $\nu_g(q, g+1) \leq 2^n$. In addition, if

$$1 \leq q < (g+3)/2 \qquad (3)$$

then $\nu_g(q, g+1) \leq 4$.

[2009] For $g = 2^{n-1}a + 1$,

$$\nu_g(q, g+1) \geq 2^n,$$

provided that $q \geq g + 1 - (g-1)/2^{n-2}$ if $n > 2$ and $q \geq 1$ for $n = 2$.

What if 2^{n-1} does not divide $g-1$, for $n>2$?

[2009] Let $g=2^{n-2}a+1$ with a odd and $n>2$. Let $q\leq g$ hold (2). Then

$$\nu_g(q,g+1)=2^n$$

if and only if $g>q>g+1-(g-1)/2^{n-2}$ or $n=3$.

The above theorem shows in particular that the lower bound for $\nu_g(q,g+1)$ cannot be improved, as for $n>3$ $g=2^{n-2}a+1$, a odd, $\nu_g(g+1-a,g+1)=\nu_g(g,g+1)=2^{n-1}$.

[2009] Let g be an odd integer. Then there exists a Riemann surface of genus g, having a pair of non-commuting fpf symmetries.

[2009] For $g = 2^k a + 1$ where $k \geq 1$ and a is odd, we have $\nu_g(g+1, g+1) \geq 2^k$ and this bound is attained for arbitrary $k \geq 2$ and $a = 1$.

[2009] Let $g, n \geq 2$ be integers such that g is odd and

$$2^n - 1 \leq g < 2^{n+1} - 1.$$

Then $\nu_g(g+1, g+1) \leq 2^n$ and the bound is attained if and only if:

1. $g = 2^{n+1} - 1 - 2^{n-l}$ and $0 \leq l < n$ or
2. $g = 2^{n+1} + 1 - 2^{n+1-l}$ for $1 \leq l < n$.