

## Mathematics of Image Processing Takehome Exam

Name: \_\_\_\_\_ Box #: \_\_\_\_\_

Due Monday, November 18 at the end of the scheduled exam.

This exam has two parts:

1. Part 1: The goal of the first part is to determine the filter coefficients for the Daubechies wavelet transform of order 2. The exercise will give us some experience at using the Fourier transform in filter design. The idea is to derive several equations from various principles and then to use the equations to solve for the filter coefficients. Some of the stuff will depend on the worksheets, and some class material.
2. Part 2: The goal of the second part is to gain some experience in using the wavelet decomposition into approximations and details by comparing compression performance for two wavelet systems, including the JPEG2000 wavelet system.

### Rules

- You can use Maple and/or Matlab, but if you do you, append the printouts and make references to them.
- Part 2 should have an explanatory 1 page memo describing the problem, what you did, and your conclusions, and appended sheets.
- If you have questions, talk to me.

## 1. The Daubechies 2 wavelets

### 4-tap filter banks

1. Let  $l_a$  and  $h_a$  be arbitrary 4-tap filter banks defined by:

$$l_a(i) = a_i, i = 0, 1, 2, 3$$

$$l_a(i) = 0, \text{ otherwise}$$

$$h_a(i) = (-1)^i a_{3-i}, i = 0, 1, 2, 3$$

$$h_a(i) = 0, \text{ otherwise}$$

Compute the 1-stage *DWT* matrix  $W_{a,1}$  on eight data points. Note that it will have entries consisting of the variables  $a_0, a_1, a_2, a_3$ . See worksheet #6.

**orthogonality relations**

2. Assume that  $W_{a,1}$  is an orthogonal matrix. This will imply certain relations among the  $a_i$ . Find all the relations.

3. Let  $V_2$  be the  $8 \times 8$  matrix obtained by passing the low pass output of the filter bank through another filter bank. It has the form:

$$\begin{bmatrix} U_4 & 0 \\ 0 & I_4 \end{bmatrix},$$

where  $U_4$  is the 1 stage DWT on four data points. Show, using calculation, that  $V_2$  is an orthogonal matrix if the orthogonality relations in 2 hold.

4. Let  $W_{a,2} = V_2 W_{a,1}$ . Show that  $W_{a,2}$  is orthogonal, based on questions 2 and 3.

## Fourier transform conditions

5. Note that the Fourier transform of an  $N$  point filter  $f$  can be written as:

$$\widehat{f}(k) = \sum_{r=0}^{N-1} f(r) \exp(-2\pi i \frac{kr}{N}).$$

Now let

$$\mathcal{P}_f(\omega) = \sum_{r=0}^{N-1} f(r) \exp(-2\pi i r\omega), 0 \leq \omega \leq 1$$

so that

$$\widehat{f}(k) = \mathcal{P}_f\left(\frac{k}{N}\right)$$

Write out the formulas for

$$\mathcal{P}_{l_a}(\omega) \text{ and } \mathcal{P}_{h_a}(\omega)$$

and then find formulas for the following quantities, and factor them (one is shown).

$$\begin{array}{ll} \mathcal{P}_{l_a}(0) = & \mathcal{P}_{h_a}(0) = \\ \mathcal{P}'_{l_a}(0) = (-2\pi i)(a_1 + 2a_2 + 3a_3) & \mathcal{P}'_{h_a}(0) = \\ \mathcal{P}''_{l_a}(0) = & \mathcal{P}''_{h_a}(0) = \end{array}$$

or alternatively find simple formulas for

$$\begin{array}{l} (-2\pi i)^{-s} \frac{d^s}{d\omega^s} \mathcal{P}_{l_a}(\omega), s = 0, 1, 2 \\ (-2\pi i)^{-s} \frac{d^s}{d\omega^s} \mathcal{P}_{h_a}(\omega), s = 0, 1, 2 \end{array}$$

6. Let  $X_s$  be the periodic signal that is defined by  $X_s(r) = r^s$  for  $-\frac{N}{2} < r < \frac{N}{2}$ , don't worry about  $X_s(\frac{N}{2})$ . Now show that

$$\begin{aligned}l_a * X_0 &= aX_0, \\l_a * X_1 &= bX_0 + cX_1, \\l_a * X_2 &= dX_0 + eX_1 + fX_2, \\h_a * X_0 &= a'X_0, \\h_a * X_1 &= b'X_0 + c'X_1, \\h_a * X_2 &= d'X_0 + e'X_1 + f'X_2,\end{aligned}$$

for constants  $a, b, c, d, e, f$  and  $a', b', c', d', e', f'$ , at least for small values of  $r$ . Find  $a, b, c, d, e, f$  and  $a', b', c', d', e', f'$  in terms of the quantities derived in question 5.

7.a Give an argument why  $\widehat{h}_a(0) = 0$ , noting it is a high pass filter.

7.b Give an argument why  $\widehat{l}_a(0) > 0$ , noting it is a low pass filter.

8. Write down all the equations from the orthogonality relations in question 1 and the linear equation from 7.a. There should be three equations, one linear and two quadratic. Show that the equations imply that

$$a_0 + a_1 + a_2 + a_3 = \sqrt{2},$$

using the equation from 7.b as necessary. (Note that for the Haar filter the corresponding equation is  $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ .) Derive an equivalent set of equations, including the one above, another linear equation and one quadratic equation.

**maximal flatness**

- 9.a Find three numerical solutions to the equation in 8. Plot the amplitudes of  $\mathcal{P}_{l_a}(\omega)$  and  $\mathcal{P}_{h_a}(\omega)$ , as a function of  $\omega$ ,  $0 \leq \omega \leq 1$ , in each case.
- 9.b Argue that the high pass filter will achieve its best high pass filtering if the graph of  $\mathcal{P}_{h_a}(\omega)$  is flat at 0. In this discussion also relate the flatness of  $\mathcal{P}_{h_a}(\omega)$  to “killing” constant and linear functions. (see question 6).

- 10.a Use the discussion in question 9 to add one more linear equations to the set of equations in question 8.
- 10.b There are four equations in four unknowns. Solve the equations. One suggestion: solve the three linear equations first to get a solution depending on only one parameter. Then plug into the quadratic equation and solve.
- 10.c Compare your result to the numerical values for the Daubechies 2 filter in Matlab.

## 2. Wavelet performance comparison

Investigate the performance of two different wavelet systems. Suggestions:

- Write a script to compute an up to 4-level wavelet decomposition and reconstruction or use the decomposition and reconstruction commands in Matlab.
- Select two wavelet systems. One should be one system from the JPEG2000 standard and the other can be any orthogonal wavelet system.
- Use two different images
- Use the following experimental compression scheme.
  - Decompose to 4 levels
  - Leave the approximation alone.
  - Pick a different thresholding percentage scheme for each of three details sub-bands.
  - Use percentage of zeros and retained energy as measures of performance.
- Vary thresholding percentages so that the image quality is still acceptable, (no perceptible degradation). Record the percentage of zeros and percentage of retained energy.
- Write up a short report on the wavelet systems you used, the threshold percentages and your conclusions.
- You may want to use the wavemenu for some experiments for the standard wavelet system.