Mathematics of Image Processing
Worksheet #8 - the 2D wavelet transform

Name:_____ Box #:_____

1-stage analysis and synthesis Let X be an image. Let $\mathcal{W}_a$ and $\mathcal{W}_s$ be the 1-stage analysis operator and synthesis operators respectively, as discussed in the notes and in class.

$$\begin{aligned} \mathcal{W}_a(X) &= W_{a,m} X W_{a,n}^t \\ \mathcal{W}_s(X) &= W_{s,m} X W_{s,n}^t \end{aligned}$$

where $W_{a,m}$, $W_{a,n}$, $W_{s,m}$, and $W_{s,n}$ are appropriately sized $1D$ analysis and synthesis matrices. Let CA, CV, CH, CD, XA, XH, XV, XD be as defined below.

$$\mathcal{W}_a(X) = \begin{bmatrix} CA & CV \\ CH & CD \end{bmatrix},$$

and

$$\begin{aligned} XA &= \mathcal{W}_s\left(\begin{bmatrix} CA & 0 \\ 0 & 0 \end{bmatrix}\right) \\ XV &= \mathcal{W}_s\left(\begin{bmatrix} 0 & CV \\ 0 & 0 \end{bmatrix}\right) \\ XH &= \mathcal{W}_s\left(\begin{bmatrix} 0 & 0 \\ CH & 0 \end{bmatrix}\right) \\ XD &= \mathcal{W}_s\left(\begin{bmatrix} 0 & 0 \\ 0 & CD \end{bmatrix}\right) \end{aligned}$$

1. Select a set of orthogonal filters l_a, h_a, l_s, h_s any way you like. Load in any image X you like. Compute CA, CV, CH, CD by the command:

   ```
   [CA,CH,CV,CD]=dwt2(X,la,ha,'mode','per')
   ```

   Show the scaled image of the matrix $\begin{bmatrix} CA & CV \\ CH & CD \end{bmatrix}$. Where does it appear that most of the energy showed up?

2. Let Z be a matrix of zeros the same size as, Compute matrices XA, XV, XH, XD by

   `XA = idwt2(CA,Z,Z,Z],ls,hs,'mode','per').`

3. Show CA and XA in the same figure window, are there any differences?

4. Verify computationally that

$$X = XA + XH + XV + XD$$

   Is there an algebraic demonstration of this fact?

5. Compute $Y \bullet Y$ for $Y = CA, CV, CH, CD, XA, XH, XV, XD$ and compare them.

6. Verify that

$$X \bullet X = XA \bullet XA + XV \bullet XV + XH \bullet XH + XD \bullet XD$$

7. Let $X_{1,d} = XV + XH + XD$ and $X_{1,a} = XA$. Plot the original, X, $X_{1,a}$ and $X_{1,d}$ and compare. How can you construct $X_{1,d}$ from a single command.

Higher stages Construct XAA, XAV, XAH, XAD from CAA, CAV, CAH, CAD using $\mathcal{W}_{s,2}$where:

$$\mathcal{W}_{a,2}(X) = \begin{bmatrix} CAA & CAV & \\ CAH & CAD & CV \\ CH & & CD \end{bmatrix},$$

8. Define the following:

$$\begin{aligned} X_{1,d} &= XV + XH + XD \text{ level 1 details} \\ X_{1,a} &= XA \text{ level 1 approximation} \\ X_{2,d} &= XAV + XAH + XAD \text{ level 2 details} \\ X_{2,a} &= XAA \text{ level 2 approximation} \end{aligned}$$

Verify that

$$\begin{aligned} X &= X_{1,a} + X_{1,d} \\ X &= X_{2,a} + X_{2,d} + X_{1,d} \end{aligned} \tag{0.1}$$

9. Do this for one more stage and compute the energies for all of detail levels and the approximations. Verify that the sums in 0.1 are orthogonal sums.