Mathematics of Image Processing
Worksheet #4 - Convolution

Name:_____ Box #:_____

Circular convolution Let f and g be two signals with N samples each.

The circular convolution of f and g is the n point sample given by:

$$f * g(k) = \sum_{r=0}^{N-1} f(k-r)g(r) = \sum_{r=0}^{N-1} f(r)g(k-r) \pmod N$$

The mod N means that we use clock arithmetic in computing $k - r$, or that we assume that f and g are defined for all integers , but are periodic with period N, i.e., $f(k+N) = f(k)$, $g(k+N) = g(k)$. For example, suppose that $N = 4$ and that f and g are defined by vectors $[x_0, x_1, x_2, x_3]$ and $[y_0, y_1, y_2, y_3]$:

$$f(r) = x_{r \bmod 4},\ g(r) = y_{r \bmod 4}.$$

Then

$$\begin{aligned}
f * g(0) &= \sum_{r=0}^{3} f(-r)g(r) = f(0)g(0) + f(-1)g(1) + f(-2)g(2) + f(-3)g(3) \\
&= f(0)g(0) + f(3)g(1) + f(2)g(2) + f(1)g(3) \\
&= x_0y_0 + x_3y_1 + x_2y_2 + x_1y_3 \\
f * g(1) &= x_1y_0 + x_0y_1 + x_3y_2 + x_2y_3 \\
f * g(2) &= x_2y_0 + x_1y_1 + x_0y_2 + x_3y_3 \\
f * g(3) &= x_3y_0 + x_2y_1 + x_1y_2 + x_0y_3
\end{aligned}$$

1. Let $N = 8$, $f(0) = \frac{1}{2}$, $f(1) = \frac{1}{2}$ $f(r) = 0$ otherwise. Let g be defined by an arbitrary vector $[y_0, y_1, \ldots, y_7]$. Find the formulas (for eight samples) for $f * g$.

2. Find a matrix H_f such that

$$\begin{bmatrix} f*g(0) \\ f*g(1) \\ \vdots \\ f*g(7) \end{bmatrix} = H_f Y = H_f \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_7 \end{bmatrix}.$$

3. Now let $N = 8$, $f(7) = x_7, f(0) = x_0, f(1) = x_1, f(r) = 0$ otherwise. Now write out the matrix for H_f.

4. Based on the examples in 3 and 4 write out the matrix for H_f for a general f.

Convolution theorem For a signal $f(n), n \in \mathbb{Z}$ which is zero for sufficiently large negative n define the z-transform by

$$F(z) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}.$$

or if $f \in \mathbb{C}^N$ is a finite signal then

$$F(z) = \sum_{n=0}^{N-1} \frac{f(n)}{z^n} = \sum_{n=0}^{N-1} f(n) z^{-n}.$$

5. Demonstrate that
$$\widehat{f}(k) = F(e^{2\pi i \frac{k}{N}}).$$

6. Let f and g be arbitrary 4 point functions defined by the vectors $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}$ and $\begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}$, respectively. Compute $h = f * g$.

7. Compute $F(z), G(z), H(z)$, and $F(z)G(z)$, Reduce the exponents of $F(z)G(z)$ mod 4 and compare to $H(z)$. What do you observe?

8. Based on previous questions give a short argument about why

$$\widehat{f * g} = \widehat{f}\widehat{g}$$

as functions.