Indiscrete Discrete Mathematics (Solutions and Comments)
(Fall 98)

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Contents

Answers and Coments to Exercises
1.5.4 - 1.8.2

Exercise 1.5.4

1. Write $(2,5,3,7)(4,6,9)$ as a product of the fewest number of transpositions possible. $(2,5)(2,3)(2,7)(4,6)(4,9)$

2. How do the number of transpositions occurring in an AT_n factorization of π and the number of transpositions in a T_n factorization of π compare? For any AT_n factorization there is always a T_n factorization which is no `longer'.

3. What can you say about the parity (evenness or oddness) of the number of transpositions in a T_n factorization of a permutation? Any two T_n facorizations of a given permutation (appear to) have the same parity. A proof appears later in the chapter (see Fact ??).

4. T F If π is the product of an even number of transpositions, then π^{-1} is a product of an even number of transpositions. T

5. T F If π is the product of an odd number of transpositions, then π^{-1} is a product of an odd number of transpositions. T

6. Determine $|T_n|$. A recursive approach yields $|T_n| = n - 1 + |T_{n-1}| = \sum_{k=1}^{n-1} k$. The multiplication principle and the observation that $(a,b) = (b,a)$ yields $\frac{n(n-1)}{2}$. Taken together these observations provide a combinatorial proof that $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$.

0.0.1 $\{(1,2),(1,2,\ldots,n)\}$

Exercise 1.5.5

1. Factor $(5,6) \in S_8$ in terms of τ and ρ. $\rho^4\tau\rho^4$

2. Factor $(5,7) \in S_8$ in terms of τ and ρ.

$$\begin{aligned}(5,7) &= (5,6)(6,7)(5,6)\\ &= \rho^4\tau\rho^4\rho^3\tau\rho^5\rho^4\tau\rho^4\\ &= \rho^4\tau\rho^7\tau\rho^9\tau\rho^4\\ &= \rho^4\tau\rho^7\tau\rho\tau\rho^4\end{aligned}$$

3. Factor $(5,6,7) \in S_8$ in terms of τ and ρ.

$$\begin{aligned}(5,6,7) &= (5,6)(5,7)\\ &= \rho^4\tau\rho^4\rho^4\tau\rho^7\tau\rho\tau\rho^4\\ &= \rho^3\tau\rho\tau\rho^4\end{aligned}$$

4. T F Each element of S_n can be factored uniquely in terms of τ and ρ. F

5. Estimate the probability that a subset of S_n consisting of just two permutations generates S_n? Sampling from S_n suggest that this probability approaches $\frac{3}{4}$ as $n \to \infty$.

```
PairsGenSn1000 := function(n);
  Sn := SymmetricGroup(n);
  Count := 0;
  for i in [1..1000] do
    Pi := Random(Sn);
    Tau := Random(Sn);
    if sub<Sn | Pi, Tau> eq Sn then
      Count : Count + 1;
    end if;
  end for;
  return Count;
end function;
print PairsGenSn1000(20);
714
```

1. T F PR_n generates S_n. T For example: We know that $\{(1,2),(1,2,3,4)\} = \{\tau,\rho\}$ generates S_4, i.e., each element of S_4 can be written as a product permutations in which each factor is either τ or ρ. Thus, if each of τ and ρ can be written as a product of elements from PR_4, then the result follows for $n = 4$. Observe that

$$\rho_{1,3,o} \cdot \rho_{2,2,o} = (1,3,2) \cdot (2,3) = (1,2) = \tau$$

and that

$$\rho^3_{1,3,i} = (1,4,3,2)^3 = (1,2,3,4) = \rho.$$

2. T F $TIAR_n$ generates S_n. T Just notice that $(1,2), (1,n,n-1,\ldots,2) \in TIAR_n$ and that $(1,n,n-1,\ldots,2)^{n-1} = (1,2,3,\ldots,n)$.

3. T F The set $\{\rho_{n,n,i}, \rho_{n,n,o}\}$ generates S_{2n}. F For $n = 4$,

$$\rho_{4,4,i} = (1,2,4,8,7,5)(3,6) \text{ and } \rho_{4,4,o} = (2,3,5)(4,7,6)$$

and Magma shows that these two permutations do not generate S_8.

```
S8 := SymmetricGroup(8);
Pi := S8 ! (1,2,4,8,7,5)(3,6);
```

```
Tau := S8 ! (2; 3; 5)(4; 7; 6);
print S8 eq sub< S8 j Pi, Tau >;
   false
```

Comment. Later in the chapter it will be clear that the reason for this result is that both of these permutations are even; i.e., they must generate a subgroup of the subgroup of even permutations.

0.1 Does f¼g generate S_n?

Exercise 1.6.1

1. Compute orders of permutations until it is second nature to you.

2. The order of a k-cycle is k.

3. List the orders which occur in S_5 (See Exercise ??):

Form	Partition	Order
(v; w; x; y; z)	[5]	5
(v; w; x; y)(z)	[4; 1]	4
(v; w; x)(y; z)	[3; 2]	6
(v; w; x)(y)(z)	$[3; 1; 1] = [3; 1^2]$	3
(v; w)(x; y)(z)	$[2; 2; 1] = [2^2; 1]$	4
(v; w)(x)(y)(z)	$[2; 1; 1; 1] = [2; 1^3]$	2
(v)(w)(x)(y)(z)	$[1; 1; 1; 1; 1] = [1^5]$	1

4. Choose ® and ¯ at random from various S_n's and compare the order of ®¯ with the orders of ® and ¯. By hand or computer, the point is that the order of a product can be poorly behaved | in particular, it is not (in general) the product of the orders.

Exercise 1.6.2

1. Check that each of these four properties hold for $\langle\pi\rangle$ where π is an arbitrary element of S_n. (Recall that each of these properties holds for S_n.) Denote $|\langle\pi\rangle|$ by k.

   Closure: $\pi^r \cdot \pi^s = \pi^{r+s} = \pi^{k \cdot i + j} = (\pi^k)^i \cdot \pi^j = \pi^j$ where $0 \le j < k$.

   Associativity: Associativity is inherited from S_n.

   Identity: $id = \pi^k = \pi^0 \in \langle\pi\rangle$.

   Inverses: $\pi^r \cdot \pi^{r-s} = id$.

2. If π is of order k, then $\pi^{-1} = \pi^h$ where $h = \underline{k-1}$.

3. T F The inverse of a permutation can always be written as a power of that permutation. T

4. Make sense out of π^{-m}. $\pi^{-m} = (\pi^{-1})^m = (\pi^m)^{-1}$

5. Under what conditions is it true that $|\langle\alpha\beta\rangle| = |\langle\alpha\rangle| \cdot |\langle\beta\rangle|$? It's true if α commutes with β and their orders are relatively prime.

6. T F $|\langle\alpha\beta\alpha^{-1}\rangle| = |\langle\alpha\rangle|$. F

7. T F $|\langle\alpha\beta\alpha^{-1}\rangle| = |\langle\beta\rangle|$. T (because $\alpha\beta\alpha^{-1}$ and β have the same FFPA form)

Exercise 1.6.3

1. T F All proper subgroups of S_n are cyclic subgroups. F $\{id, (1,2), (3,4), (1,2)(3,4)\}$ is a subgroup of S_4 which is not cyclic.

2. Exhibit an element, say π_{52}, in S_{52} of maximum order.

$$\begin{aligned}\pi_{52} = \; & (1;2;3;4;5;6;7;8;9;10;11;12;13) \cdot (14;15;16;17;18;19;20;21;22;23;2 \\ & \cdot(25;26;27;28;29;30;31;32;33) \cdot (34;35;36;37;38;39;40) \\ & \cdot(41;42;43;44;45) \cdot (46;47;48;49) \cdot (50;51;52)\end{aligned}$$

$$|\langle\pi_{52}\rangle| = \mathrm{lcm}[13;11;9;7;5;4;3] = 180;180:$$

Comment. π_{52} is not unique nor is the associated FFPA form. The question of just how many permutations induce the same FFPA form is dealt with in Chapter 2. Some students might want some empirical help to get started on this:

```
> pi := Random(SymmetricGroup(52));
> print pi, Order(pi);
```

3. Compute $|\langle\pi_{52}\rangle| = |S_{52}|$: $\frac{180180}{52!} \doteq 2{:}2339 \times 10^{-63}$

4. Make a conjecture concerning

$$\lim_{n\to\infty} \frac{|\langle\pi_n\rangle|}{n!} \qquad = 0:$$

0.2 A shuffling status report

Exercise 1.7.1

Discuss the practicality of shuffling an 52-card deck using each of the following generating sets.

1. S_{52}: No | this is n-card pickup.

2. CYC_{52}: No.

3. T_{52}: No.
4. AT_{52}: Maybe.
5. $\{(1,2),(1,2,3,\ldots,n-1,n)\}$: Yes.
6. $TIAR_{52}$: Yes.
7. PR_{52}: No, but standard shuffling is our attempt at a reasonable approximation.

   Comment. These answers are assuming `practicality' means that a human being might be able to perform the shuffles with a deck prior to a game of cards.

0.3 Directed labelled graphs

Exercise 1.8.1

1. T F AT_n will shuffle an n-card deck. F
2. Exhibit E_4 and O_4 for the Cayley digraph of S_4 induced by AT_4.

E_4	$O_4 = E_4 \cdot (1,2)$
id	$(1,2)$
$(1,2,3)$	$(2,3)$
$(1,3,2)$	$(1,3)$
$(1,3,4)$	$(1,3,4,2)$
$(1,4,3)$	$(1,4,3,2)$
$(1,2,4)$	$(2,4)$
$(1,4,2)$	$(1,4)$
$(2,3,4)$	$(1,2,3,4)$
$(2,4,3)$	$(1,2,4,3)$
$(1,2)(3,4)$	$(3,4)$
$(1,3)(2,4)$	$(1,3,2,4)$
$(1,4)(2,3)$	$(1,4,2,3)$

3. Draw the Cayley digraph of S_3 induced by T_3.

abc bac

cab acb

bca cba

4. T F T_3 will shu²e a 3-card deck. F

5. T F T_n will shu²e an n-card deck. F

6. Draw the Cayley digraph of S_3 induced by $TIAR_3$.

abc bac

cab acb

bca cba

7. T F $TIAR_3$ will shu²e a 3-card deck. T (or so it appears)

8. T F $TIAR_n$ will shu²e an n-card deck. T (or so it appears)

Exercise 1.8.2

1. Draw the card position digraph of N_3 induced by T_3.
   ® = (1; 2); ¯ = (2; 3); ° = (1; 3)

1 2 3

2. Draw the card position digraph of N_3 induced by $TIAR_3$.
   i = id; ¾ = (1; 2); ¯ = (1; 3; 2)

   1 2 3

3. Draw the card position digraph of N_4 induced by AT_4.
   ® = (1; 2); ¯ = (2; 3); ± = (3; 4)

   1 2 3 4

4. Draw the card position digraph of N_4 induced by T_4.
   ® = (1; 2); ¯ = (2; 3); ± = (3; 4); ° = (1; 3); " = (1; 4); ! = (2; 4)

   1 2 3 4

5. Draw the card position digraph of N_4 induced by $TIAR_4$.
   i = id; ¾ = (1; 2); ¿ = (1; 3; 2); ¯ = (1; 4; 3; 2)

   1 2 3 4