Answers and Comments to Exercises

Section 1.4

Exercise 1.4.2

1. Determine the number of association schemes for evaluating a product of four permutations.

   ® ¢ (¯ ¢ (° ¢ ±))
   ® ¢ ((¯ ¢ °) ¢ ±)
   (® ¢ ¯) ¢ (° ¢ ±)   5
   (® ¢ (¯ ¢ °)) ¢ ±
   ((® ¢ ¯) ¢ °) ¢ ±

2. Determine the number of association schemes for evaluating a product of ¯ve permutations. Let a_n denote the number of association schemes for a product of $n + 1$ permuations (i.e., n multiplication symbols are used). We have that

   $$a_0 = 1; a_1 = 1; a_2 = 2; a_3 = 5$$

   by listing and counting the possibilities. Listing is productive when you get to $n = 4$ if you recognize that the association schemes can be organized using the `naked' multiplication symbol that must appear in each association scheme.

   ® ¢̤ (¯ ¢ ° ¢ ± ¢ ") $ ® ¢ (¯ ¢ (° ¢ (± ¢ ")))
   ® ¢ (¯ ¢ ((° ¢ ±) ¢ "))
   ® ¢ ((¯ ¢ °) ¢ (± ¢ "))   5 = a_0 ¢ a_3
   ® ¢ ((¯ ¢ (° ¢ ±)) ¢ ")
   ® ¢ (((¯ ¢ °) ¢ ±) ¢ ")

   (® ¢ ¯) ¢̤ (° ¢ ± ¢ ") $ (® ¢ ¯) ¢ (° ¢ (± ¢ "))
   (® ¢ ¯) ¢ ((° ¢ ±) ¢ ")   2 = a_1 ¢ a_2

   (® ¢ ¯ ¢ °) ¢̤ (± ¢ ") $ (® ¢ (¯ ¢ °)) ¢ (± ¢ ")
   ((® ¢ ¯) ¢ °) ¢ (± ¢ ")   2 = a_2 ¢ a_1

$$(\text{®}\,¢\,\text{¯}\,¢\,\text{°}\,¢\,\text{±})\,\underset{"}{¢}\,\text{"}\ \ \$\ \ \left.\begin{array}{l} (\text{®}\,¢\,(\text{¯}\,¢\,(\text{°}\,¢\,\text{±})))\,¢\,\text{"} \\ (\text{®}\,¢\,((\text{¯}\,¢\,\text{°})\,¢\,\text{±}))\,¢\,\text{"} \\ ((\text{®}\,¢\,\text{¯})\,¢\,(\text{°}\,¢\,\text{±}))\,¢\,\text{"} \\ ((\text{®}\,¢\,(\text{¯}\,¢\,\text{°}))\,¢\,\text{±})\,¢\,\text{"} \\ (((\text{®}\,¢\,\text{¯})\,¢\,\text{°})\,¢\,\text{±})\,¢\,\text{"} \end{array}\right\}\ 5 = a_0\,¢\,a_3$$

$$a_4 = 14 = 5+2+2+4 = a_0¢a_3 + a_1¢a_2 + a_2¢a_1 + a_3¢a_0 = \sum_{i=0}^{3} a_i a_{3\,¡\,i}$$

3. Can you generalize the previous two counts to a product of n permutations?

$$a_{n+1} = \sum_{i=0} a_i\,¢\,a_{n\,¡\,i}$$

Exercise 1.4.3

1. Evaluate µ ¢ id: µ
2. Evaluate id ¢ µ: µ

Exercise 1.4.4

1. Exhibit two cycles, say $\tilde{A}_1$ and $\tilde{A}_2$, in S_8 which do not satisfy

$$\tilde{A}_1\tilde{A}_2 = \tilde{A}_1\tilde{A}_2.$$

$$(1;2)(1;3) = (1;2;3) \neq (1;3;2) = (1;3)(1;2)$$

2. What is it about ® = (1; 4) and ¯ = (2; 7; 5; 8; 3) that guarantees

$$\text{®}\,¢\,\text{¯} = \text{¯}\,¢\,\text{®}?$$

The cycles are disjoint (so they don't `interact').

3. Complete a commutativity matrix for S_3: Put a 1 (0) at the intersection of the ¾-th row and ¿-th column of the following array if ¾¿ = ¿¾ (¾¿ 6= ¿¾).

	id	(1; 2; 3)	(1; 3; 2)	(1; 2)	(2; 3)	(1; 3)
id	1	1	1	1	1	1
(1; 2; 3)	1	1	1	0	0	0
(1; 3; 2)	1	1	1	0	0	0
(1; 2)	1	0	0	1	0	0
(2; 3)	1	0	0	0	1	0
(1; 3)	1	0	0	0	0	1

4. Determine the probability that two elements of S_3 commute; i.e., the probability that ¾¿ = ¿¾ for ¾; ¿ 2 S_3: $\frac{18}{36} = \frac{6¢3}{6¢6} = \frac{3}{6}$

5. Complete a commutativity matrix for S_4 and determine the probablility that two elements of S_4 commute. $\frac{120}{576} = \frac{24¢5}{24¢24} = \frac{5}{24}$

6. Determine the probablility that two elements of $S_5; S_6$; and S_7 commute.

```
CommPairs := function(n);
  Count := 0;
  for Pi in SymmetricGroup(n) do
    for Tau in SymmetricGroup(n) do
```

1.

```
      if Pi*Tau eq Tau*Pi then
        Count := Count + 1;
      end if;
    end for;
  end for;
  return Count;
end function;

 print CommPairs(5);
         840
```

$\frac{840}{(5!)^2} = \frac{5!¢7}{(5!)^2} = \frac{7}{5!}$

```
print CommPairs(6);
7920
```

$$\frac{7920}{(6!)^2} = \frac{6! \cdot 11}{(6!)^2} = \frac{11}{6!}$$

```
print CommPairs(7);
75600
```

$$\frac{75600}{(7!)^2} = \frac{7! \cdot 15}{(7!)^2} = \frac{15}{7!}$$

Comment. These probabilities are structured to suggest that there is a pattern waiting to be discovered. In particular we will eventually recognize that through $n = 7$ the probabilites are the ratio of the number of partitions of n to the cardinality of S_n. This observation will finally be explained in terms of Burnside's Lemma.

2. Estimate the probability that two elements of S_8 and S_{52} commute.

```
EstCommPairs1000:= function(n);
   Count := 0;
   for i in [1..1000] do
      Pi := Random(SymmetricGroup(n));
      Tau := Random(SymmetricGroup(n));
         if Pi*Tau eq Tau*Pi then
            Count := Count + 1;
         end if;
```

1.
```
      end for;
      return Count;
   end function;
```

```
print EstCommPairs1000(8);
1
```

Estimated probability $\frac{1}{1000}$

```
print EstCommPairs1000(52);
0
```

Estimated probability $\frac{0}{1000}$

2. How does the probability that two elements of S_n commute behave as $n \to \infty$? The following Magma code

```
for n in [1..100] do
   print EstCommPairs1000(n);
end for;
```

should convince the students the probabiliity is very close to zero.

Exercise 1.4.5

The process of getting dressed after your morning shower is a composition of several dressing `factors'. Two of these factors are a (putting on your socks) and b (putting on your shoes). Check to see that

$$ab \neq ba:$$

Comment. Walking into the classroom with a sock over one of your shoes makes this point quite clear.

Exercise 1.4.6

1. Use the FFPA to represent elements of S_n until the process is second nature to you. Here are a few in S_9 to get you started:

   (a) $(2;3;7)(1;3;9;7)(2;7;5) = (1;3)(2;9;5)$

   (b) $(1;5)(5;6)(2;7;9)(5;4) = (1;6;4;5)(2;7;9)$

   (c) $(2;7;5;3;4)(8;9)(1;5;2;6)(8;9) = (1;5;3;4;6)(2;7)$

   (d) $(1;2)(1;3)(1;4)(1;5)(1;6)(1;7)(1;8)(1;9) = (1;2;3;4;5;6;7;8;9)$

2. The FFPA `form' of a permutation in S_n induces a natural partition of n because each element of N_n occurs in one, and only one, cycle of the permutation. For example, the form $(v;w;x)(y;z)$,

which is the `same' as the form $(v;w)(x;y;z)$, induces the partition $[3;2]$ of five. Complete the following table for S_5.

Form	Cardinality	Partition of 5
$(v;w;x;y;z)$	$\frac{5!}{5} = 4! = 24$	$[5]$
$(v;w;x;y)(z)$	$\frac{5\cdot4\cdot3\cdot2}{4} = 30$	$[4;1]$
$(v;w;x)(y;z)$	$\frac{5\cdot4\cdot3}{3} \cdot \frac{2\cdot1}{2} = 20$	$[3;2]$
$(v;w;x)(y)(z)$	$\frac{5\cdot4\cdot3}{3} \cdot \frac{2\cdot1}{2!} = 20$	$[3;1;1] = [3;1^2]$
$(v;w)(x;y)(z)$	$\frac{\frac{5\cdot4}{2} \cdot \frac{3\cdot2}{2}}{2} = 15$	$[2;2;1] = [2^2;1]$
$(v;w)(x)(y)(z)$	$\frac{5\cdot4}{2} \cdot \frac{3\cdot2\cdot1}{3!} = 10$	$[2;1;1;1] = [2;1^3]$
$(v)(w)(x)(y)(z)$	$\frac{5\cdot4\cdot3\cdot2\cdot1}{5!} = 1$	$[1;1;1;1;1] = [1^5]$

Comment. An equivalence relation on permutations is rearing its head without waiting for a formal definition. This is also a good time to emphasize structured counting, even though counting techniques have yet to be developed.

3. Complete the analogous table for S_4:

Form	Cardinality	Partition of 4
$(v;w;x;y)$	$\frac{4!}{4} = 3! = 6$	$[4]$
$(v;w;x)(y)$	$\frac{4\cdot3\cdot2}{3} = 8$	$[3;1]$
$(v;w)(x;y)$	$\frac{\frac{4\cdot3}{2} \cdot \frac{2\cdot1}{2}}{2} = 3$	$[2;2] = [2^2]$
$(v;w)(x)(y)$	$\frac{4\cdot3}{2} \cdot \frac{2\cdot1}{2!} = 6$	$[2;1;1] = [2;1^2]$
$(v)(w)(x)(y)$	$\frac{4\cdot3\cdot2\cdot1}{4!} = 1$	$[1;1;1;1] = [1^4]$

4. Why are the forms $(v;w;x)(y;z)$ and $(v;w)(x;y;z)$ the same? The cycles are disjoint so $(v;w)(x;y;z) = (x;y;z)(v;w)$; i.e., both forms look like $(a;b;c)(d;e)$:

5. Which is larger (and why), the number of partitions of n or $n!$? The larger is $n!$ because each partition of n corresponds (in a natural way via the FFPA) to one or more permutations.

6. Let $\pi = (1;6;3)(2;7;5;4) \in S_7$. Record the successive powers of

¼ (¼; ¼ ¢ ¼ = ¼^{2}; ¼ ¢ ¼ ¢ ¼ = ¼^{3}; etc.) in FFPA form.

¼ = (1; 6; 3)(2; 7; 5; 4)
¼^{2} = (1; 3; 6)(2; 5)(4; 7)
¼^{3} = (2; 4; 5; 7)
¼^{4} = (1; 6; 3)
¼^{5} = (1; 3; 6)(2; 7; 5; 4)
¼^{6} = (2; 5)(4; 7)
¼^{7} = (1; 6; 3)(2; 4; 5; 7)
¼^{8} = (1; 3; 6)
¼^{9} = (2; 7; 5; 4)
¼^{10} = (1; 6; 3)(2; 5)(4; 7)
¼^{11} = (1; 3; 6)(2; 4; 5; 7)
¼^{12} = id
¼^{13} = (1; 6; 3)(2; 7; 5; 4)
¼^{14} = (1; 3; 6)(2; 5)(4; 7)
¼^{15} = (2; 4; 5; 7)

Comment. After the students produce this data by hand I give them the relevant Magma code:

```
S7 := SymmetricGroup(7);
Pi := S7!(1,6,3)(2,7,5,4);
for n in [1..15] do
   print n, Pi^n;
end for;
```

1. Let #1-cycles(¼) denote the number of 1-cycles (¯xed points) appearing in the FFPA form of ¼:

   (a) Determine the minimum value of #1-cycles(¼) for ¼ 2 S_{10}: 0

   (b) Determine the maximum value of #1-cycles(¼) for ¼ 2 S_{10}: 10

   (c) Estimate the average value of #1-cycles(¼) for ¼ 2 S_{10}: ¼ 1

   (d) Generalize to S_n: 0; n; 1

2. Let #2-cycles(¼) denote the number of 2-cycles (transpositions) appearing in the FFPA form of ¼:

(a) Determine the minimum value of $\#2\text{-cycles}(\pi)$ for $\pi \in S_{10}$: 0

(b) Determine the maximum value of $\#2\text{-cycles}(\pi)$ for $\pi \in S_{10}$: 5

(c) Estimate the average value of $\#2\text{-cycles}(\pi)$ for $\pi \in S_{10}$: $\approx \frac{1}{2}$

(d) Generalize to S_n: $0; \left\lfloor \frac{n}{2} \right\rfloor; \frac{1}{2}$

3. Generalize your observations in 1 and 2 to $\#k\text{-cycles}(\pi)$ in S_n: $0; \left\lfloor \frac{n}{k} \right\rfloor; \approx \frac{1}{k}$

4. Let $\#\text{cycles}(\pi)$ denote the number of disjoint cycles appearing in the FFPA form of π. (Don't forget that each fixed point of π contributes one to $\#\text{cycles}(\pi)$.)

(a) Determine the minimum value of $\#\text{cycles}(\pi)$ for $\pi \in S_{10}$: 1

(b) Determine the maximum value of $\#\text{cycles}(\pi)$ for $\pi \in S_{10}$: 10

(c) Estimate the average value of $\#\text{cycles}(\pi)$ for $\pi \in S_{10}$: $\approx \sum_{k=1}^{10} \frac{1}{k} = \frac{7381}{2520} \approx 2.929$. For those students who have had calculus it is worth a quick geometrical discussion of why this estimate is trying to look like $\ln 10 \approx 2.303$

(d) Generalize to S_n: $1; n; \sum_{k=1}^{n} \frac{1}{k} \approx \ln n$. **Comment.** I do the previous four items as a classroom discussion exercise by partitioning the class into teams (two or three students per team), providing each team with a random sample of 50 pemutations from S_{10} using Magma (for i in [1..500] do print Random(SymmetricGroup(10)); end for;), and recording the relevant counts on the board in the following format.

Sample	#1-cycles	#2-cycles	#3-cycles	#4-cycles	#5-cycles	#cycles
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
Total						
Average						

Discussion ensues.

5. Compute the following products:

   (a) (1; 2; 3; 4; 5)(6; 7; 8)(9; 10)¢(2; 7) = (1; 7; 8; 6; 2; 3; 4; 5)(9; 10)

   (b) (1; 2; 3; 4; 5; 6; 7; 8)(9; 10)¢(2; 7) = (1; 7; 8)(2; 3; 4; 5; 6)(9; 10)

6. Let ¼ be written in FFPA form and let ¿ be a transposition. Generate data until you can write a formula for #cycles(¼¿) in terms of #cycles(¼):Start by computing the following two products.

   (a) (1; 2; 3; 4; 5)(6; 7; 8)(9; 10)¢(2; 7): = (1; 7; 8; 6; 2; 3; 4; 5)(9; 10)

   (b) (1; 2; 3; 4; 5; 6; 7; 8)(9; 10)¢(2; 7): = (1; 7; 8)(2; 3; 4; 5; 6)(9; 10)

$$\#\text{cycles}(¼¿) = \begin{cases} \#\text{cycles}(¼)\ ¡\ 1 & \text{if the symbols of ¿ occur in di®erent cycles of ¼:} \\ \#\text{cycles}(¼) + 1 & \text{if the symbols of ¿ occur in a common cycles of ¼:} \end{cases}$$

   Comment. This result is used later in the chapter to prove that the even permutations of S_n form a subgroup of S_n.