Answers and Comments to Exercises

1.1.1, 1.1.2, 1.1.3, 1.2.1,1.3.1,1.3.2,1.3.3

Exercise 1.1.1

1. Verify that several is 60. 6 ¢ 2 ¢ 5 = 60

2. Determine the number of disjoint-cycle names for;

   (a) µ $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 6 & 1 & 8 & 2 & 4 & 3 \end{matrix}$ ¶ $ (1; 5; 8; 3; 6; 2; 7; 4)) 8

   (b) 2; 6; 5; 7; 8; 1; 4; 3 $ (1; 6; 2)(3; 8; 5)(4; 7)) 6 ¢ 3 ¢ 3 ¢ 2 = 108

Comment. This innocuous exercise is pointing towards the multiplication principle

Exercise 1.1.2

1. Exhibit our favorite name for;

   (a) µ $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 6 & 1 & 8 & 2 & 4 & 3 \end{matrix}$ ¶ $ (1; 5; 8; 3; 6; 2; 3; 4)

   (b) 2; 6; 5; 7; 8; 1; 4; 3 $ (1; 6; 2)(3; 8; 5)(4; 7)

2. Become comfortable with array, list, and disjoint-cycle notation by completing (and if necessary, extending) the following table.

	disjoint-cycle	array	list
½	(1; 8; 6; 3; 7)(2; 4)	µ $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 7 & 2 & 5 & 3 & 1 & 6 \end{matrix}$ ¶	7; 4; 6; 2; 5; 8; 3; 1
¾	(1; 3)(2; 7; 5; 8)	µ $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 4 & 8 & 6 & 5 & 2 \end{matrix}$ ¶	3; 8; 1; 4; 7; 6; 2; 5
¿	(1; 8; 2; 5; 3)	µ $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 1 & 4 & 3 & 6 & 7 & 2 \end{matrix}$ ¶	3; 8; 5; 4; 2; 6; 7; 1

Exercise 1.1.3

1. Read the definition of group and then give at least one (mathematical) reason why you can't refer to $\{\pi, \rho, \sigma, \tau\}$ as a group. $\{\pi, \rho, \sigma, \tau\}$ doesn't contain the identity permutation (or it isn't closed or inverses are missing).

2. Determine the cardinality of S_8:

$$|S_8| = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8! = 40{,}320$$

3. An adjacency occurs in $\pi \in S_n$ if

$$\ldots, i, i+1, \ldots \quad \text{or} \quad \ldots, i+1, i, \ldots$$

appears in the list notation for π. For example, three adjacencies occur in

$$2, 6, 5, 7, 8, 1, 4, 3 \in S_8.$$

Exercise 1.2.1

1. Determine the cardinality of S_1, S_2, S_3, S_4, S_5, S_6, and S_7. 1, 2, 6, 24, 120, 720, and 40,320 respectively

2. Determine a formula for the cardinality of S_n. $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 := n!$

3. Determine a formula for the number of functions from a set of cardinality n to a set of cardinality k.

$$\underset{1}{\underline{k}} \cdot \underset{2}{\underline{k}} \cdot \cdots \cdot \underset{n}{\underline{k}} = k^n$$

4. Determine a formula for the number of n letter 'words' that can be constructed using an alphabet consisting of k letters.

$$\underset{1}{\underline{k}} \cdot \underset{2}{\underline{k}} \cdot \cdots \cdot \underset{n}{\underline{k}} = k^n$$

Comment. Items 3 and 4 of this exercise provide an opportunity to reinforce the value of looking at examples before trying to state general results and to illustrate that 'different' problems are often mathematically equivalent.

$$
\begin{array}{ccc}
\text{Domain} & \xrightarrow{f} & \text{Codomain} \\
 & & \\
1 & & a \\
2 & & b \\
3 & & c \\
4 & &
\end{array}
\qquad
\leftrightarrow
\begin{array}{cccc}
b & a & c & a \\
\uparrow & \uparrow & \uparrow & \uparrow \\
1 & 2 & 3 & 4
\end{array}
\leftrightarrow \text{baca}
$$

The correspondence between functions and words is used throughout the text.

Exercise 1.3.1

1. Give the disjoint-cycle name for the following 8-card shuffles.

   (a) $\rho_{4,4,o} = (2,3,5)(4,7,6)$

   (b) $\rho_{6,2,i} = (5,6,8,7)$

2. Give the disjoint-cycle name for the following 10-card shuffles.

   (a) $\rho_{5,5,i} = (1,2,4,8,5,10,9,7,3,6)$

   (b) $\rho_{5,5,o} = (2,3,5,9,8,6)(4,7)$

3. Estimate $|PR_n|$: A crude application of the multiplication principle suggests $\underbrace{(n+1)}_{\text{cuts}} \cdot \underbrace{2}_{i,o} = 2n+2$. Recognition of the fact that $\rho_{n,0,i}$, $\rho_{n,0,o}$, $\rho_{0,n,i}$, $\rho_{0,n,o}$, and $\rho_{n-1,1,o}$ all produce the identity permutation finally yields $2n-2 = \underbrace{(n-1)}_{\text{nontrivial cuts}} \cdot \underbrace{2}_{i,o}$

Exercise 1.3.2

1. Apply $½_{3;5;o}$ repeatedly to a deck and record the resulting `products' (i.e., $½_{3;5;o} ¢ ½_{3;5;o}$; $½_{3;5;o} ¢ ½_{3;5;o} ¢ ½_{3;5;o}$; and so on).

$$\begin{aligned}
½_{3;5;o} &= (1;3;7;6;4)(2;5)\\
½^{2}_{3;5;o} &= (1;7;4;3;6)\\
½^{3}_{3;5;o} &= (1;6;3;4;7)(2;5)\\
&\vdots\\
½^{10}_{3;5;o} &= \mathrm{id}\\
½^{11}_{3;5;o} &= ½_{3;5;o}
\end{aligned}$$

2. Do the same thing with $½_{4;4;i}$.

$$\begin{aligned}
½_{4;4;i} &= (1;2;4;8;7;5)(3;6)\\
½^{2}_{4;4;i} &= (1;4;7)(2;8;5)\\
½^{3}_{4;4;i} &= (1;8)(2;7)(5;5)(3;6)\\
&\vdots\\
½^{6}_{4;4;i} &= \mathrm{id}\\
½^{7}_{4;4;i} &= ½_{4;4;i}
\end{aligned}$$

   Comment. The ¯rst two items of this exercise provide an opportunity to hint at the eventual role of modular arithmetic in permutation algebra (and eventually group theory).

3. Give the disjoint-cycle name for $½_{4;4;i} ¢ ½_{3;5;o}$:

$$(1;2;4;8;7;5)(3;6) ¢ (1;3;7;6;4)(2;5) = (1;5;3;4;8;6;7;2)$$

   Comment. Some relevant Magma code:

```
>S8 := SymmetricGroup(8);
>RHO44i := S8!(1,2,4,8,7,5)(3,6);
>RHO35o := S8!(1,3,7,6,4)(2,5);
>print RHO35o;
    (1,3,7,6,4)(2,5)
>print RHO35o^5;
    (2,5)
>print RHO44i*RHO35o;
    (1,5,3,4,8,6,7,2)
```

4. T F The `product' of two PRSs is another PRS. **False**: the PRSs of S_3 are id, $(1;2); (2;3)$; and $(1;3;2)$ and the non-PRSs are $(1;2;3)$ and $(1;3)$: Notice that $(1;2) \cdot (1;3) = (1;2;3)$: It is also interesting to note that the fact that $(1;3;2) \cdot (1;3;2) = (1;2;3)$ generalizes to show that $\rho^2_{1;n-1;1;i}$ can not be a PRS because it moves the top two cards to the bottom of the deck and any PRS with $n > 2$ that moves the second card to the bottom of the deck will separate the top two cards. (Another argument? If a deck can be shuffled using PRSs and PR_n is `closed' under permutation multiplication, then $n! \quad 2n - 2$ — recall Exercise)

Exercise 1.3.3

1. Apply $\tau_{5;8}$, repeatedly, and record the resulting `products'.

$$\begin{aligned}
\tau_{5;8} &= (1;5;4;3;2) \\
\tau^2_{5;8} &= (1;4;2;5;3) \\
\tau^3_{5;8} &= (1;3;5;2;4) \\
&\vdots \\
\tau^5_{5;8} &= \mathrm{id} \\
\tau^6_{5;8} &= \tau_{5;8}
\end{aligned}$$

2. Do the same thing with $\tau_{7;8}$:

$$\begin{aligned}
\tau_{7;8} &= (1;7;6;5;4;3;2) \\
\tau^2_{7;8} &= (1;6;4;2;6;5;3) \\
\tau^3_{7;8} &= (1;5;2;6;3;7;4) \\
&\vdots \\
\tau^5_{7;8} &= \mathrm{id} \\
\tau^6_{7;8} &= \tau_{7;8}
\end{aligned}$$

3. Give the disjoint-cycle name for $\tau_{5;8} \cdot \tau_{7;8}$:

$$(1;5;4;3;2) \cdot (1;7;6;5;4;3;2) = (1;4;2;7;6;5;3)$$

4. Give the disjoint-cycle name for $\tau_{7;8} \cdot \tau_{5;8}$:

$$(1;7;6;5;4;3;2) \cdot (1;5;4;3;2) = (1;7;6;4;2;5;3)$$

5. Determine j$TIAR_n$j: n

6. T F The `product' of two TIARs is another TIAR. False: for n > 2 the list notation for ¿$^2_{n;n}$ is 3; 4; 5; : : : ; n; 1; 2 while the list notation for a TIAR begins with 1 (the identity) or 2 (any non-identitiy TIAR). (Another argument? If a deck can be shu²ed using TIARSs and $TIAR_n$ is `closed' under permutation multiplication, then n! n:)