ECE-597: Probability, Random Processes, and Estimation

Homework # 4

Due: Friday April 15, 2016

1) Assume $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{Y}$ are jointly Gaussain with

$$\underline{\mu}_X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, K_{\mathbf{XX}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & 4 \\ 3 & 4 & 15 \end{bmatrix}, \ \underline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{Y} \end{bmatrix}$$

a) Determine the optimal estimate of $\mathbf{Y}$ based on observations of $\mathbf{X}_1$ and $\mathbf{X}_2$ using the *orthogonality principle*. To do this, you first need to create zero mean random variables,

$$\underline{\mathbf{V}}_{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_1 - \mu_1 \\ \mathbf{X}_2 - \mu_2 \end{bmatrix} \qquad \mathbf{V}_{\mathbf{Y}} = \mathbf{Y} - \mu_3$$

Then $\hat{\mathbf{V}}_{\mathbf{Y}} = \underline{a}^T \underline{\mathbf{V}}_{\mathbf{X}}$. The orthogonality principle is then

$$E[(\mathbf{V}_{\mathbf{Y}} - \hat{\mathbf{V}}_{\mathbf{Y}})\underline{\mathbf{V}}_{\mathbf{X}}^T] = E[(\mathbf{V}_{\mathbf{Y}} - \underline{a}^T \underline{\mathbf{V}}_{\mathbf{X}})\underline{\mathbf{V}}_{\mathbf{X}}^T] = 0;$$

You can find the needed covariance matrices from $K_{\mathbf{XX}}$ above.
Answer: $\hat{\mathbf{Y}} = 3.67\mathbf{X}_1 - 0.33\mathbf{X}_2$.

2) The optimal linear estimator of $\mathbf{X}$ given observation $\mathbf{Y}$ was derived in class to be

$$\hat{\mathbf{X}}(\mathbf{Y}) = E[\mathbf{X}] + \frac{COV(\mathbf{X}, \mathbf{Y})}{\sigma_{\mathbf{Y}}^2}(\mathbf{Y} - E[\mathbf{Y}])$$

a) Derive the Mean Squared Error relationship for the optimal linear estimator of $\mathbf{X}$ given $\mathbf{Y}$

$$E[(\mathbf{X} - \hat{\mathbf{X}})^2] = \sigma_{\mathbf{X}}^2 - COV(\mathbf{X}, \mathbf{Y})^2/\sigma_{\mathbf{Y}}^2$$

b) Derive the normalized MSE relationship

$$E[(\frac{\hat{\mathbf{X}} - \mathbf{X}}{\sigma_{\mathbf{X}}})^2] = 1 - \rho_{\mathbf{XY}}^2$$

3) Assume

$$\mathbf{Y} = h\mathbf{X} + \mathbf{V}$$

where $\mathbf{X}$ is the random variable we want to estimate, $\mathbf{Y}$ is the random variable we observe, $\mathbf{V}$ is observation noise which is uncorrelated with $\mathbf{X}$ and $\mathbf{Y}$. We have showed that the optimal linear estimator of $\mathbf{X}$ given $\mathbf{Y}$ is

$$\hat{\mathbf{X}}(\mathbf{Y}) = E[\mathbf{X}] + \frac{h\sigma_{\mathbf{X}}^2}{h^2\sigma_{\mathbf{X}}^2 + \sigma_{\mathbf{V}}^2}(\mathbf{Y} - E[\mathbf{Y}])$$

Now assume $E[\mathbf{X}] \neq 0$, $E[\mathbf{V}] = 0$, and $E[\mathbf{XV}] = 0$. We want to determine the optimal linear estimator using the orthogonality principle. For this we need zero mean variables. Let

$$\begin{aligned}\mathbf{W} &= \mathbf{Y} - \mu_{\mathbf{Y}} \\ \mathbf{U} &= \mathbf{X} - \mu_{\mathbf{X}}\end{aligned}$$

a) Use the orthogonality principle to determine the optimal c such that

$$\hat{\mathbf{U}}(\mathbf{W}) = c\mathbf{W}$$

that is, we want

$$E[(\mathbf{U} - \hat{\mathbf{U}})\mathbf{W}] = 0$$

b) Show that your answer reduces to the form we derived previously, i.e. go back to $\mathbf{X}$ and $\mathbf{Y}$.

4) Given the joint density

$$f_{\mathbf{X},\mathbf{Y}}(x, y) = x + y$$

for $0 \leq x \leq 1, 0 \leq y \leq 1$, $E[\mathbf{X}] = E[\mathbf{Y}] = 0.583$, $\sigma^2_{\mathbf{X}} = \sigma^2_{\mathbf{Y}} = 0.07639$, and $E[\mathbf{XY}] = 0.333333$

a) Determine the optimal linear estimator of $\mathbf{X}$ given $\mathbf{Y}$ is observed.
b) Determine the optimal estimator of $\mathbf{X}$ given $\mathbf{Y}$ is observed.
c) If $\mathbf{Y} = 0$ and 1, what are the estimates of X using the two estimators?

5) Consider N observations $\mathbf{R}_1, \mathbf{R}_2, ...\mathbf{R}_N$. Under both H_1 and H_0 the $\mathbf{R}_i$ are independent and identically distributed zero mean Gaussian random variables. Under H_1 each $\mathbf{R}_i$ has variance σ_1^2, while under H_0 each $\mathbf{R}_i$ has variance σ_0^2. Show that for $\sigma_1^2 > \sigma_0^2$ the likelihood ratio test becomes

$$\sum_{i=1}^{i=N} r_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}\left(\ln\eta - N\ln\frac{\sigma_0}{\sigma_1}\right)$$

6) Under hypothesis H_0 random variable $\mathbf{X}$ has the density $f_{\mathbf{X}|H_0}$, while under hypothesis H_1 it has the density $f_{\mathbf{X}|H_1}$. The *a priori* probabilities and densities are shown below:

$$f_{\mathbf{X}|H_0} = \frac{2}{\sqrt{2\pi}}e^{-\frac{(x-1)^2}{2}}u(x) \qquad P(H_0) = \frac{1}{3}$$

$$f_{\mathbf{X}|H_1} = e^{-x}u(x) \qquad P(H_1) = \frac{2}{3}$$

Assume costs $c_{01} = c_{10} = 1$ and $c_{11} = c_{00} = 0$. Using the likelihood ration test, determine the decision regions and construct a simple graph indicating which hypothesis will be accepted for all real observed values of random variable $\mathbf{X}$.

7) The Poisson distribution of events is encountered frequently as a model of shot noise and other diverse phenomenon. Each time the experiment is conducted a certain number of events occur. Our observation is just this number which ranges from 0 to ∞ and obeys a Posson distribution on both hypotheses; that is

$$P(N = n) = \frac{(m_i)^n}{n!} e^{-m_i}, \; n = 0, 1, 2, ..., \; i = 0, 1$$

and m_i is that parameter that specifies the average number of events: $E(n) = m_i$. Show that the likelihood ratio test is

$$n \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln \eta + m_1 - m_0}{\ln m_1 - \ln m_0} \quad \text{for } m_1 > m_0$$

and

$$n \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\ln \eta + m_1 - m_0}{\ln m_1 - \ln m_0} \quad \text{for } m_0 > m_1$$

8) Assume we have Gaussian distributed two random vectors with means $\underline{\mu}_0$ and $\underline{\mu}_1$ and covariance matrices K_0 andK_1.
a) Show that for the observed random vector $\underline{\mathbf{R}}$, the likelihood ratio test is

$$\frac{1}{2}(\underline{r} - \underline{\mu}_0)^T K_0^{-1} (\underline{r} - \underline{\mu}_0) - \frac{1}{2}(\underline{r} - \underline{\mu}_1)^T K_1^{-1} (\underline{r} - \underline{\mu}_1) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta + \frac{1}{2} \ln |K_1| - \frac{1}{2} \ln |K_0|$$

Note that this decision is between two quadratic forms.

b) If $K_1 = K_2 = K$ (equal covariance matrices), show that the above result reduces to

$$(\underline{\mu}_1 - \underline{\mu}_0)^T K^{-1} r \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta + \frac{1}{2}(\underline{\mu}_1^T K^{-1} \underline{\mu}_1 - \underline{\mu}_0^T K^{-1} \underline{\mu}_0)$$