ECE-597: Probability, Random Processes, and Estimation

Homework # 3

Due: Tuesday March 29, 2016
Exam 1: Thursday March 31, 2016

1) Show that if $\mu = E[\mathbf{X}_i], i = 1, 2, \ldots, n$ is known then

$$\hat{\Theta} = \frac{1}{n}\sum_{i=1}^{i=n}(\mathbf{X}_i - \mu)^2$$

is unbiased for estimating σ^2. The $\mathbf{X}_i$ are i.i.d. random variables with $Var(\mathbf{X}_i) = \sigma^2$.

2) Consider the random variable $\mathbf{X}$ that satisfies the binomial law, that is

$$P(\mathbf{X} = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

a) Show that

$$\hat{\Theta} = \frac{\mathbf{X}}{n}$$

is unbiased and for p.

b) Show that

$$\hat{\Theta} = \frac{\mathbf{X}(\mathbf{X}-1)}{n(n-1)}$$

is unbiased for p^2.

For this problem (Problem 2) you should explicitly evaluate the expected values, do not just look them up since you should know how to do this.

3) Given the Gaussian density

$$f_{\mathbf{X}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

with n i.i.d. observations
a) Compute the maximum likelihood estimates (MLE) of μ and σ^2.
b) Prove equations 6.7-10 and 6.7-11 by working out the details.
c) What are the MLE of μ and σ^2 when $n = 1$? What conclusions can you draw from this result?

4) Let $\mathbf{Z}_i = [\mathbf{X_i},\ \mathbf{Y}_i]^T, i = 1, 2, \ldots, n$ be n i.i.d. observations with

$$f_{\mathbf{Z}}(z) = \frac{1}{2\pi(1-\rho^2)^{1/2}\sigma_1\sigma_2} e^{-Q(z)}$$

where $z = (x, y)^T$ and

$$Q(z) \;=\; \frac{1}{2(1-\rho^2)}\left[\left(\frac{x}{\sigma_1}\right)^2 - 2\rho\frac{xy}{\sigma_1\sigma_2} + \left(\frac{y}{\sigma_2}\right)^2\right]$$

Assume $\sigma_1, \sigma_2 > 0$, $|\rho| < 1$. Computer the MLE of ρ assuming σ_1 and σ_2 are known.

Hint: Since you assume both σ_1 and σ_2 are known, set $n = \sum(x_i/\sigma_1)^2 = \sum(y_i/\sigma_2)^2$. The correct answer is $\hat{\rho} = \frac{1}{n}\sum(x_i y_i/\sigma_1\sigma_2)$.

5) An important thing to know is the **Cramer-Rao** bound for estimators. This provides a lower bound on the variance of the estimator.

Theorem If $\hat{\Theta}$ is *any* unbiased estimate of parameter θ, then

$$Var(\hat{\Theta}) \geq \left(E\left\{\left[\frac{\partial \ln f_{x|\theta}(x|\theta)}{\partial\theta}\right]^2\right\}\right)^{-1}$$

or, equivalantly,

$$Var(\hat{\Theta}) \geq \left\{-E\left[\frac{\partial^2 \ln f_{x|\theta}(x|\theta)}{\partial^2\theta}\right]\right\}^{-1}$$

assuming the partial derivatives both exist and are absolutely integrable. Any estimator which satisfies this bound with an equality is said to be *efficient*.

For estimating the parameter p for a binomial distribution using n observations we get the unbiased estimator

$$\hat{\Theta} = X/n$$

Show that this estimator is efficient using the Theorem above.

Hint: $Var(\hat{\Theta}) = p(1-p)/n$, $E[\mathbf{X}] = np$, $E[\mathbf{X}^2] = np(1-p) + n^2p^2$. You can just use these relationships, you don't have to derive them in this problem.