ECE 597: Probability, Random Processes, and Estimation
Homework # 1

Due: Tuesday March 15, 2016

From the book: 4.10, 4.20, 4.50, 4.51, 4.74

Hints and Answers:
4.10, feel free to use Table 4.3.2. This is a very short problem.
4.20, feel free to use what you know about Gaussian random variables, this is a short problem.
4.50, the answer is

$$R_n(k) = \begin{cases} (1+a^2)\sigma^2 & k=0 \\ -a\sigma^2 & k=\pm 1 \\ 0 & \text{else} \end{cases}$$

4.51, start with $R_n(0)$, then $R_n(1)$, etc. The answer is $R_n(l) = b^{|l|}K$
4.74, you should get $COV(\epsilon, X) = \alpha\sigma_x^2 - \sigma_x\sigma_y\rho_{xy}$ and if you put in the optimal values the covariance should be zero.

Additional Problems

In the problems that follow you might find it useful to remember

$$\rho_{AB} = \frac{E[(\mathbf{A}-\mu_a)(\mathbf{B}-\mu_b)]}{\sigma_a\sigma_b} = \frac{COV(\mathbf{A},\mathbf{B})}{\sigma_a\sigma_b}$$

1) Assume $\mathbf{X}$ is a discrete random variable, with

$$P(\mathbf{X}=-1) = \frac{1}{2},\ P(\mathbf{X}=0) = \frac{1}{4},\ P(\mathbf{X}=1) = \frac{1}{4}$$

and the random variable $\mathbf{Y}$ is defined in terms of the random variable $\mathbf{X}$ as

$$\mathbf{Y} = 2\cos(\pi\mathbf{X})$$

a) Find the mean and variance of $\mathbf{X}$
b) Find the mean and variance of $\mathbf{Y}$

Answers: -1/4, 11/16, -1, 3

2) Assume $\mathbf{X}$ is a discrete random variable, with

$$P(\mathbf{X}=0) = \frac{1}{2},\ P(\mathbf{X}=1) = \frac{1}{4},\ P(\mathbf{X}=2) = \frac{1}{4}$$

and the random variable $\mathbf{Y}$ is defined in terms of the random variable $\mathbf{X}$ as

$$\mathbf{Y} = (\mathbf{X}-2)^2$$

a) Find the mean and variance of $\mathbf{X}$
b) Find the mean and variance of $\mathbf{Y}$

Answers: 3/4, 11/16, 9/4, 51/16

3) For random variables $\mathbf{X}$ and $\mathbf{Y}$ we know $\mu_x = 1$, $\mu_y = 2$, $\sigma_x^2 = 4$, $\sigma_y^2 = 1$, and $\rho_{xy} = 0.4$. Now we make up two new random variables

$$\begin{aligned}\mathbf{V} &= -\mathbf{X} + 2\mathbf{Y} \\ \mathbf{W} &= \mathbf{X} + 3\mathbf{Y}\end{aligned}$$

a) Show that $\mu_v = 3$ and $\mu_w = 7$
b) Show that $\sigma_v^2 = 4.8$ and $\sigma_w^2 = 17.8$.
c) Show that the covariance $COV(\mathbf{W}, \mathbf{V}) = 1.2$
d) Show that $\rho_{wv} = 0.13$

4) For random variables $\mathbf{X}$ and $\mathbf{Y}$ we know $\mu_x = 0$, $\mu_y = 0$, $\sigma_x^2 = 4$, $\sigma_y^2 = 16$, and $\rho_{xy} = -0.5$. Now we make up the random variable

$$\mathbf{W} = (a\mathbf{X} + 3\mathbf{Y})^2$$

a) Determine a symbolic expression for the value of a that minimizes μ_w.
b) Show that $\mu_w = 108$.

5) Assume we have random variables $\mathbf{Y} = a\mathbf{X} + b$, where $\mathbf{X}$ is normally distributed (Gaussian) with mean zero and variance one. Determine the parameters a and b so $\mathbf{Y}$ will be normally distributed with a mean of μ and a variance of σ^2.

6) Recall that for a random variable Z with Gaussian pdf, we have

$$F_Z(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{1}{2\sigma_z^2}(z-\mu_z)^2}$$

and the mean of Z is μ_z and the variance of Z is σ_z^2.

Starting with the joint pdf for two Gaussian random variables X and Y (equation 4.3-27 in the text), and assuming that $\mathbf{Y}$ is Gausian, show that

$$\begin{aligned}VAR[X|Y=y] &= \sigma_x^2(1-\rho_{xy}^2) \\ E[X|Y=y] &= \mu_x + \frac{\rho_{xy}\sigma_x}{\sigma_y}(y-\mu_y)\end{aligned}$$

Hint: Write the conditional pdf $f_{X|Y}(x|y) = Ae^{-B}$ and solve for A and B, keeping in mind the general form for a Gaussian pdf. You do not need to do any integration, only algebra. *We will use this result in the next computer project.*