Name: Solutions

ECE 597: Probability, Random Processes, and Estimation
Exam #2
Friday May 6, 2016
Calculators may only be used for simple calculations.

Possible Useful Equations

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]w[n-k] \\
\sum_{m=-\infty}^{\infty} \gamma^m u[m]u[N-m] &= \frac{1-\gamma^{N+1}}{1-\gamma} \\
Z\left\{b^{|l|}\right\} &= \frac{1-b^2}{(1-bz^{-1})(1-bz)} \\
Z\left\{x[n-k]\right\} &= z^{-k}X(z) \\
S_{\mathbf{XX}}(z) &= K_o H_c(z) H_c(z^{-1}) \\
H_{Wiener}(z) &= \frac{1}{K_o}\frac{1}{H_c(z)}\left[\frac{S_{dx}(z)}{H_c(z^{-1})}\right]_+ \\
\epsilon^2 &= R_{dd}[0] - \sum_{m=0}^{\infty} h[m]R_{dx}[m]
\end{aligned}
$$

$$
\frac{f_{R|H_1}(r|H_1)}{f_{R|H_0}(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0(C_{10}-C_{00})}{P_1(C_{01}-C_{11})}
$$

You may assume all random variables and sequences are real valued.

1) Assume $X[n]$ is an i.i.d random sequence taking on values 1 and 0, with

$$P\{\mathbf{X}[n]=1\} = \frac{1}{3}$$
$$P\{\mathbf{X}[\mathbf{n}]=\mathbf{0}\} = \frac{2}{3}$$

Define then a new random sequence $\mathbf{Z}[n]$, where

$$\mathbf{Z}[n] = \sum_{k=1}^{k=n} \mathbf{X}[k]$$

Determine $\mu_{\mathbf{X}}[n]$, $\mu_{\mathbf{Z}}[n]$, $R_{\mathbf{XX}}[n,m]$, and $R_{\mathbf{ZZ}}[n,m]$

$$E\{X[n]\} = 1\cdot\frac{1}{3} + 0\cdot\frac{2}{3} = \boxed{\frac{1}{3} = \mu_x[n]}$$

$$E\{Z[n]\} = E\left\{\sum_{k=1}^{n} X[k]\right\} = \sum_{k=1}^{n} E\{X[k]\} = \sum_{k=1}^{n} \frac{1}{3} = \boxed{\frac{n}{3} = \mu_z[n]}$$

$$R_{XX}[n,m] = E\{X[n]\,X[m]\}$$

if $n \neq m$, since i.i.d. $R_{XX}[n,m] = \mu_x[n]\,\mu_x[m] = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$

if $n = m$ $R_{XX}[n,n] = E\{X[n]^2\} = 1^2\cdot\frac{1}{3} + 0^2\cdot\frac{2}{3} = \frac{1}{3}$

$$\boxed{R_{xx}[n,m] = \frac{1}{9} + \frac{2}{9}\,\delta[n-m]}$$

$$R_{zz}[n,m] = E\left\{\sum_{k=1}^{n}\sum_{l=1}^{m} X[k]\,X[l]\right\} = \sum_{k=1}^{n}\sum_{l=1}^{m} E\{X[k]\,X[l]\}$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{m} R_{XX}[k,l] = \sum_{k=1}^{n}\sum_{l=1}^{m}\left(\frac{1}{9} + \frac{2}{9}\,\delta[n-m]\right)$$

$$= \boxed{\frac{nm}{9} + \frac{2}{9}\min(n,m) = R_{zz}[n,m]}$$

2) Consider a signal $S[n]$ with additive white noise $\mathbf{V}[n]$, so the input to the system is

$$\mathbf{X}[n] = S[n] + \mathbf{V}[n]$$

Assume $\mathbf{V}[n]$ is a zero mean process which is uncorrelated with $S[n]$. The correlations $R_{SS}[n]$ and $R_{\mathbf{VV}}[n]$ are given as

$$\begin{aligned} R_{SS}[n] &= 0.3^{|n|} \\ R_{\mathbf{VV}}[n] &= 0.5\delta[n] \end{aligned}$$

Determine the optimal linear (causal and stable) estimator for $S[n+1]$ using the Wiener filter. You can assume

$$S_{\mathbf{XX}}(z) = \frac{1.440(1-0.1042z)(1-0.1042z^{-1})}{(1-0.3z)(1-0.3z^{-1})}$$

$$K_0 = 1.440 \qquad H_c(z) = \frac{1-0.1042z^{-1}}{1-0.3z^{-1}}$$

$$R_{dx}[n] = E\{d[k]\,X[k-n]\} = E\{S[k+1]\,(S[k-n] + V[k-n])\} = R_{ss}[n+1]$$

$$S_{dx}(z) = S_{ss}(z) = z\,\frac{1-0.3^2}{(1-0.3z)(1-0.3z^{-1})} = \frac{(0.91)z}{(1-0.3z)(1-0.3z^{-1})}$$

$$\left[\frac{S_{dx}(z)}{H_c(z^{-1})}\right] = [G(z)] = \frac{z(0.91)}{(1-0.3z)(1-0.3z^{-1})}\cdot\frac{1-0.3z}{1-0.1042z} = \frac{z(0.91)}{(1-0.3z^{-1})(1-0.1042z)}$$

$$= \frac{z(0.91)}{z^{-1}(z-0.3)(-0.1042)(z-9.597)} = \frac{-8.733z^2}{(z-0.3)(z-9.597)}$$

$$\frac{G(z)}{z} = \frac{-8.733z}{(z-0.3)(z-9.597)} = \frac{A}{z-0.3} + \frac{B}{z-9.597} \qquad A = \frac{(-8.733)(0.3)}{0.3-9.597} = 0.282$$

$$\left[\frac{S_{dx}(z)}{H_c(z^{-1})}\right]_+ = \frac{0.282}{1-0.3z^{-1}} \qquad H(z) = \frac{1}{1.440}\left[\frac{1-0.3z^{-1}}{1-0.1042z^{-1}}\right]\left[\frac{0.282}{1-0.3z^{-1}}\right]$$

$$= \frac{0.196}{1-0.1042z^{-1}} \qquad \boxed{h[n] = 0.196(0.1042)^n u[n]}$$

3) Assume we have the zero mean WSS random sequence $\mathbf{X}[n]$. Assume we want to estimate $\mathbf{Y}[n]$ using the estimator

$$\hat{Y}[n] = a\mathbf{X}[n] + b\mathbf{X}[n-1]$$

Determine a matrix equation to be solved to determine a and b using the principle of orthogonality (the error is proportional to the data used to make the estimator). Your matrix equation will be in terms of $R_{\mathbf{XX}}$ and $R_{\mathbf{YX}}$.

$$E\left\{\left(Y[n] - \hat{Y}[n]\right)X[n]\right\} = E\left\{\left(Y[n] - aX[n] - bX[n-1]\right)X[n]\right\} = 0$$

$$R_{YX}[0] - a R_{XX}[0] - b R_{XX}[-1] = 0$$

$$E\left\{\left(Y[n] - \hat{Y}[n]\right)X[n-1]\right\} = E\left\{\left(Y[n] - aX[n] - bX[n-1]\right)X[n-1]\right\} = 0$$

$$R_{YX}[1] - a R_{XX}[1] - b R_{XX}[0] = 0$$

$$\boxed{\begin{bmatrix} R_{XX}[0] & R_{XX}[-1] \\ R_{XX}[1] & R_{XX}[0] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} R_{YX}[0] \\ R_{YX}[1] \end{bmatrix}}$$

4) Consider a signal $S[n]$ with additive white noise $\mathbf{V}[n]$, so the input to the system is

$$\mathbf{X}[n] = S[n] + \mathbf{V}[n]$$

where

- $S[n] = a^n u[n]$ is a **deterministic, not random** signal
- $\mathbf{V}[n]$ is zero mean white noise with $R_{\mathbf{VV}}[n,m] = \sigma^2\delta[n-m]$
- the impulse response of the system is given by $h[n] = b^n u[n]$.

Determine the expected output power $E\{\mathbf{Y}^2[n]\}$

Hints:

- Assume you use the form $y[n] = \sum h[k]x[n-k]$
- You will need to be sure to use two sums with two dummy indices
- It is probably easier if you write $R_{\mathbf{XX}}[n,m]$ instead of using a single argument

$$E\{Y[n]^2\} = E\left\{\sum_{k=-\infty}^{\infty} h[k]\,X[n-k] \sum_{m=-\infty}^{\infty} h[m]\,X[n-m]\right\}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} h[k]\,h[m]\,R_{XX}[n-k,n-m]$$

$$R_{XX}[n-k,n-m] = E\{(S[n-k]+V[n-k])(S[n-m]+V[n-m])\}$$

$$= E\{S[n-k]\,S[n-m]\} + E\{V[n-k]\,V[n-m]\}$$

$$= a^{n-k}u[n-k]\,a^{n-m}u[n-m] + R_{VV}[n-k,n-m]$$

$$= a^{n-k}u[n-k]\,a^{n-m}u[n-m] + \sigma^2\delta[m-k]$$

$$E\{Y[n]^2\} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left\{b^k u[k]\,b^m u[m]\,a^{n-k}u[n-k]\,a^{n-m}u[n-m] + b^k u[k]\,b^m u[m]\,\sigma^2\delta[m-k]\right\}$$

$$= \sum_{k=0}^{n} a^n\left(\frac{b}{a}\right)^k \sum_{m=0}^{n} a^n\left(\frac{b}{a}\right)^m + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} b^k b^m \sigma^2\delta[m-k]$$

$$= \left(a^n\left[\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{b}{a}}\right]\right)^2 + \sum_{k=0}^{\infty}(b^2)^k\sigma^2 = \boxed{\left(a^n\left[\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{b}{a}}\right]\right)^2 + \frac{\sigma^2}{1-b^2}}$$