Name: Solutions

Problem 1 ________/10
Problem 2 ________/20
Problem 3 ________/10
Problem 4 ________/25
Problem 5 ________/25
Problem 6 ________/10

ECE 597: Probability, Random Processes, and Estimation
Exam #1

Thursday March 31, 2016

No calculators or computers allowed.

1) (10 points)) Assume we have a random variable X that takes on the values 1 and 0 with the probabilities

$$P(\mathbf{X}=1) = \frac{2}{3}$$
$$P(\mathbf{X}=0) = \frac{1}{3}$$

We now construct the new random variable Y as

$$\mathbf{Y} = 2\mathbf{X}+3$$

Determine $\mu_{\mathbf{Y}}$ and $\sigma^2_{\mathbf{Y}}$

$P(Y=5) = 2/3 \qquad P(Y=3) = 1/3$

$\mu_Y = 5 \cdot 2/3 + 3 \cdot 1/3 = \frac{10}{3} + \frac{3}{3} = \boxed{\frac{13}{3} = \mu_Y}$

$E[Y^2] = 5^2 \cdot 2/3 + 3^2 \cdot 1/3 = \frac{50}{3} + \frac{9}{3} = \frac{59}{3}$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = \boxed{\frac{59}{3} - \left(\frac{13}{3}\right)^2 = \sigma_Y^2}$

2) (20 points) Assume we have the joint density

$$f_{X,Y}(x,y) = 2ye^{-x} \quad 0 \le y \le 1, \ 0 \le x < \infty$$

Note: The ranges of X and Y are different

a) Determine the marginal density $f_X(x)$
b) Determine the marginal density $f_Y(y)$
c) Are X and Y independent? Why or why not?
d) Determine $E[Y|X]$

a) $f_X(x) = \int_0^1 2ye^{-x}dy = e^{-x}\int_0^1 2y\,dy = e^{-x}\left[y^2\Big|_0^1\right] = \boxed{e^{-x} = f_X(x)}$

b) $f_Y(y) = \int_0^\infty 2ye^{-x}dx = 2y\int_0^\infty e^{-x}dx = 2y\left[-e^{-x}\Big|_0^\infty\right] = \boxed{2y = f_Y(y)}$

c) $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ so independent

d) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y)$

$E[Y|X] = \int_0^1 y\,f_{Y|X}(y|x)\,dy = \int_0^1 y(2y)\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}y^3\Big|_0^1 = 2/3$

$\boxed{E[Y|X] = 2/3}$

3) (10 points) Assume $\underline{X}$ and $\underline{W}$ are random vectors, not necessarily of the same size. Assume also that K_{XX}, K_{WW} and K_{WX} are known. Now we make a new random vector

$$\underline{Y} = A\underline{X} + B\underline{W} + \underline{C}$$

where A and B are constant matrices (not necessarily of the same size), and $\underline{C}$ is a constant vector. Determine an expression for K_{YY} in terms of these known quantities ONLY. Do not assume the means are zero.

Hint: $(FG)^T = G^T F^T$

$$\mu_y = A\underline{\mu_x} + B\underline{\mu_w} + \underline{c}$$

$$K_{yy} = E\left[(\underline{Y} - \underline{\mu_y})(\underline{Y} - \mu_w)^T\right]$$

$$= E\left[(A\underline{X} + B\underline{W} + \underline{C} - A\underline{\mu_x} - B\underline{\mu_w} - c)(A\underline{X} + B\underline{W} + \underline{c} - A\underline{\mu_x} - B\underline{\mu_w} - c)^T\right]$$

$$= E\left[\{A(\underline{X} - \underline{\mu_x}) + B(\underline{W} - \underline{\mu_w})\}\{A(\underline{X} - \underline{\mu_x}) + B(\underline{W} - \underline{\mu_w})\}^T\right]$$

$$= E\left[A(\underline{X} - \underline{\mu_x})(\underline{X} - \underline{\mu_x})^T A^T\right] + E\left[A(\underline{X} - \underline{\mu_x})(\underline{W} - \underline{\mu_w})^T B^T\right]$$

$$+ E\left[B(\underline{W} - \mu_w)(\underline{X} - \underline{\mu_x})^T A^T\right] + E\left[B(\underline{W} - \underline{\mu_w})(\underline{W} - \mu_w)^T B^T\right]$$

$$= A K_{xx} A^T + A K_{xw} B^T + B K_{wx} A^T + B K_{ww} B^T$$

$$K_{xw} = K_{wx}^T$$

$$\boxed{K_{YY} = A K_{xx} A^T + A K_{wx}^T B^T + B K_{wx} A^T + B K_{ww} B^T}$$

The two formulas may (or may not) be useful in the following problem.

The general formula for a multidimensional Gaussian density is

$$f_{\underline{\mathbf{X}}}(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\left[det(K_{\mathbf{XX}})^{\frac{1}{2}}\right]}\; exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T K_{\mathbf{XX}}^{-1}(\underline{x}-\underline{\mu})\right\}$$

The inverse of a 2 x 2 matrix is given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4) (25 points) Assume the random vector $\underline{X} = [X_1\ X_2]^T$ has the Gaussian density given by

$$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi\sqrt{1}}\, exp\left\{-\frac{1}{2}\left[(x_1-1)^2 - 2(x_1-1)x_2 + 2x_2^2\right]\right\}$$

Determine $\underline{\mu}$ and K_{XX}

by inspection $\boxed{\underline{\mu} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$$[x_1\ x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2]\begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix} = ax_1^2 + bx_1x_2 + bx_2x_1 + cx_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2 = x_1^2 - 2x_1x_2 + 2x_2^2$$

$$a = 1 \quad b = -1 \quad c = 2$$

$$K_{XX}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad K_{XX} = \frac{1}{2-1}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = K_{XX}}$$

5) (25 points) Assume $\underline{X}$ is a zero mean random Gaussian vector with covariance

$$K_{XX} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

Assume we form the random vector Y using the transformation

$$Y = AX$$

where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

a) Determine the resulting marginal pdf's, $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$.
b) If the mean value of X had been $\mu_X = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, determine the resulting mean value of Y

$$K_{YY} = A K_{XX} A^T$$

$$B = A K_{XX} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ -4 & 2 \end{bmatrix}$$

$$K_{YY} = B A^T = \begin{bmatrix} 7 & 14 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 35 & 0 \\ 0 & 10 \end{bmatrix}$$

or

$$B = K_{XX} A^T = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 14 & 2 \end{bmatrix}$$

$$K_{YY} = AB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & -4 \\ 14 & 2 \end{bmatrix} = \begin{bmatrix} 35 & 0 \\ 0 & 10 \end{bmatrix}$$

$$K_{YY}^{-1} = \frac{1}{350}\begin{bmatrix} 10 & 0 \\ 0 & 35 \end{bmatrix} = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$$

$$\mu_Y = A\mu_X = 0$$

$$f_Y = \frac{1}{2\pi\sqrt{350}} e^{-\frac{1}{2}[y_1 \; y_2]\begin{bmatrix} 1/35 & 0 \\ 0 & 1/10 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \frac{1}{2\pi\sqrt{350}} e^{-\frac{1}{2}y_1^2/35} e^{-\frac{1}{2}y_2^2/10}$$

$$\boxed{\begin{aligned} f_{Y_1}(y_1) &= \frac{1}{\sqrt{2\pi\cdot 35}} e^{-\frac{1}{2}y_1^2/35} \\ f_{Y_2}(y_2) &= \frac{1}{\sqrt{2\pi\cdot 10}} e^{-\frac{1}{2}y_2^2/10} \end{aligned}}$$

6) (10 points) Assume we have an experiment where the random variable X is assumed to follow a *uniform* density, i.e.,

$$f_X(x) = \frac{1}{\theta} \quad 0 < x \leq \theta$$

Assume we preform the experiment n times with outcomes $x_1, x_2, \ldots, x_n$, What is the maximum liklihood estimate of θ based on these observations?

Hint: Taking derivatives here will not help, you are going to have to think about this. You don't need to really do any math to come up with the answer.

$L(\theta) = \frac{1}{\theta^n}$

for observation x_1, the minimum of $\frac{1}{\theta}$ would be $\frac{1}{x_1}$

x_2, - - - - - - $\frac{1}{x_2}$

so for the n observations, the minimum value of θ is $\boxed{\max(x_1, \ldots, x_n)}$