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ECE 597: Probability, Random Processes, and Estimation
Exam #1

Thursday March 31, 2016

No calculators or computers allowed.

1) **(10 points)) Assume we have a random variable $\mathbf{X}$ that takes on the values 1 and 0 with the probabilities**

$$P(\mathbf{X}=1) \;=\; \frac{2}{3}$$
$$P(\mathbf{X}=0) \;=\; \frac{1}{3}$$

We now construct the new random variable $\mathbf{Y}$ as

$$\mathbf{Y} \;=\; 2\mathbf{X}+3$$

Determine $\mu_{\mathbf{Y}}$ and $\sigma^2_{\mathbf{Y}}$

2) (20 points) Assume we have the joint density

$$f_{\mathbf{X},\mathbf{Y}}(x,y) \quad = \quad 2ye^{-x} \quad 0 \leq y \leq 1,\ 0 \leq x < \infty$$

Note: The ranges of X and Y are different

a) Determine the marginal density $f_{\mathbf{X}}(x)$
b) Determine the marginal density $f_{\mathbf{Y}}(y)$
c) Are X and Y independent? Why or why not?
d) Determine $E[\mathbf{Y}|\mathbf{X}]$

3) (10 points) Assume $\underline{\mathbf{X}}$ and $\underline{\mathbf{W}}$ are random vectors, not necessarily of the same size. Assume also that $K_{\mathbf{XX}}$, $K_{\mathbf{WW}}$ and $K_{\mathbf{WX}}$ are known. Now we make a new random vector

$$\underline{\mathbf{Y}} \;=\; A\underline{\mathbf{X}} + B\underline{\mathbf{W}} + \underline{C}$$

where A and B are constant matrices (not necessarily of the same size), and $\underline{C}$ is a constant vector. Determine an expression for $K_{\mathbf{YY}}$ in terms of these known quantities ONLY. Do not assume the means are zero.

Hint: $(FG)^T = G^T F^T$

The two formulas may (or may not) be useful in the following problem.

The general formula for a multidimensional Gaussian density is

$$f_{\underline{\mathbf{X}}}(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\left[det(K_{\mathbf{XX}})^{\frac{1}{2}}\right]} \; exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T K_{\mathbf{XX}}^{-1}(\underline{x}-\underline{\mu})\right\}$$

The inverse of a 2 x 2 matrix is given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4) (25 points) Assume the random vector $\underline{\mathbf{X}} = [\mathbf{X}_1\ \mathbf{X}_2]^T$ has the Gaussian density given by

$$f_{\underline{\mathbf{X}}}(\underline{x}) \quad = \quad \frac{1}{2\pi\sqrt{1}}\ exp\left\{-\frac{1}{2}\left[(x_1-1)^2 - 2(x_1-1)x_2 + 2x_2^2\right]\right\}$$

Determine $\underline{\mu}$ and $K_{\mathbf{XX}}$

5) (25 points) Assume $\underline{\mathbf{X}}$ is a zero mean random Gaussian vector with covariance

$$K_{\mathbf{XX}} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

Assume we form the random vector Y using the transformation

$$\mathbf{Y} = A\mathbf{X}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

a) Determine the resulting marginal pdf's, $f_{\mathbf{Y_1}}(y_1)$ and $f_{\mathbf{Y_2}}(y_2)$.
b) If the mean value of X had been $\mu_{\mathbf{X}} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, determine the resulting mean value of Y

6) (10 points) Assume we have an experiment where the random variable X is assumed to follow a *uniform* density, i.e.,

$$f_{\mathbf{X}}(x) = \frac{1}{\theta} \quad 0 < x \leq \theta$$

Assume we preform the experiment n times with outcomes $x_1, x_2, \ldots, x_n$, What is the maximum liklihood estimate of θ based on these observations?

Hint: Taking derivatives here will not help, you are going to have to think about this. You don't need to really do any math to come up with the answer.