Name:

ECE 597: Probability, Random Processes, and Estimation

Exam #2

Friday May 8, 2015

Calculators may only be used for simple calculations.

1) (15 points) Assume $X[n]$ is an i.i.d random sequence taking on values 1 and 0, with

$$P\{X[n]=1\} = \frac{1}{3}$$
$$P\{X[n]=0\} = \frac{2}{3}$$

Determine $\mu_{\mathbf{X}}[n]$ and $R_{\mathbf{XX}}[n,m]$

2) (15 points) Under hypothesis H_0, random variable $\mathbf{X}$ has density $f_{\mathbf{X}|H_0}$, while under hypothesis H_1, random variable $\mathbf{X}$ has density $f_{\mathbf{X}|H_1}$. The *a priori* probabilities and densities are

$$f_{\mathbf{X}|H_0}(x|H_0) = \frac{2}{\sqrt{2\pi}}e^{-\frac{(x-1)^2}{2}}u(x), \; P(H_0) = \frac{1}{3}$$
$$f_{\mathbf{X}|H_1}(x|H_1) = e^{-x}u(x), \; P(H_1) = \frac{2}{3}$$

Assume costs $c_{01} = c_{10} = 1$ and $c_{00} = c_{11} = 0$. Using the likelihood ratio test, determine the decision regions and construct a simple graph indicating which hypothesis will be accepted for all real observed values of the random variable $\mathbf{X}$.

3) (15 points) Consider a signal $S[n]$ with additive white noise $V[n]$, so the input to the system is

$$X[n] = S[n] + V[n]$$

Assume $V[n]$ is a zero mean process which is uncorrelated with $S[n]$, and assume both $V[n]$ and $S[n]$ are jointly WSS.

The output of the system is given by $Y[n] = \sum_{k=-\infty}^{\infty} h[k]X[n-k]$.

Determine an expression for $R_{\mathbf{YX}}[m]$ in terms of $h[n]$, $R_{\mathbf{SS}}$ and $R_{\mathbf{VV}}$.

4) (55 points) Consider a signal $S[n]$ with additive white noise $V[n]$, so the input to the system is

$$X[n] = S[n] + V[n]$$

Assume $V[n]$ is a zero mean process which is uncorrelated with $S[n]$, and assume both $V[n]$ and $S[n]$ are jointly WSS. Finally, assume

$$\begin{aligned} R_{\mathbf{SS}}[n] &= 0.9^{|n|} \\ R_{\mathbf{VV}}[n] &= 0.2\delta[n] \end{aligned}$$

a) (20 points) Assume we want to estimate $S[n]$ using the first two values of the observed signal,

$$\hat{d}[n] = h_o X[n] + h_1 X[n-1]$$

Using the principle of orthogonality (the error is orthogonal to the data used in the estimate), to determine a system of equations to be solved to determine h_0 and h_1. For full credit, the system of equations should be all numbers except for the unknowns h_0 and h_1. If you do this correctly, you should get, $h_0 \approx 0.616$ and $h_1 \approx 0.286$, but you do not need to solve the system of equations.

b) (20 points) Now assume we are going to construct the optimal linear (causal and stable) estimator for $S[n]$ using the Wiener filter,

$$\begin{aligned} \hat{d}[n] &= \sum_{k=0}^{k=\infty} h_o[k] X[n-k] \\ H_o(z) &= \frac{1}{K_o} \frac{1}{H_c(z)} \left[\frac{S_{dx}(z)}{H_c(z^{-1})} \right]_+ \end{aligned}$$

where you can use

$$\begin{aligned} a^{|m|} &\leftrightarrow \frac{1-a^2}{(1-az^{-1})(1-az)} \\ S_{\mathbf{XX}}(z) &= \frac{0.4822(1-0.371z)(1-0.371z^{-1})}{(1-0.9z^{-1})(1-0.9z)} \end{aligned}$$

The mean squared error for the optimal filter is approximately 0.113, but you do not need to compute this.

c) (15 points) For your solution to part (a), compute the mean squared error

$$MSE = E\left\{(S[n] - \hat{d}[n])^2\right\}$$

Hint: this is easy to simplify if you use the orthogonality principle and remember that $\hat{d}[n]$ is a linear function of the data used in making the estimator. You answer here should be a number. You should use the values for h_0 and h_1 you were given in part a.