Name: Solutions

ECE 597: Probability, Random Processes, and Estimation

Exam #1

Thursday April 2, 2015

1) Assume we have the joint density

$$f_{\mathbf{X},\mathbf{Y}}(x,y) = \frac{1}{3}(xy+1) \quad 0 \le x \le 1,\ 0 \le y \le 2$$

Note: The ranges of **X** and **Y** are different
a) Determine the marginal density $f_{\mathbf{X}}(x)$
b) Determine the marginal density $f_{\mathbf{Y}}(y)$
c) Are **X** and **Y** independent? Why or why not?
d) Determine $E[\mathbf{X}|\mathbf{Y}]$ (note that this will be a function of y)

a) $f_X(x) = \int_0^2 f_{X,Y}(x,y)\,dy = \frac{1}{3}\int_0^2 (xy+1)\,dy = \frac{1}{3}\left[x\frac{y^2}{2}+y\right]\Big|_0^2 = \boxed{\frac{1}{3}\left[2x+2\right] = f_X(x)}$

b) $f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx = \frac{1}{3}\int_0^1 (xy+1)\,dx = \frac{1}{3}\left[\frac{x^2}{2}y+x\right]\Big|_0^1 = \boxed{\frac{1}{3}\left[\frac{y}{2}+1\right] = f_Y(y)}$

c) $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$ so not independent

d) $E[X|Y] = \int_0^1 x f_{X|Y}(x|y)\,dx = \int_0^1 x \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx = \int_0^1 \frac{x\frac{1}{3}(xy+1)}{\frac{1}{3}(\frac{y}{2}+1)}\,dx$

$= \frac{1}{\frac{y}{2}+1}\int_0^1 (x^2y+x)\,dx = \frac{1}{\frac{y}{2}+1}\left(\frac{x^3}{3}y+\frac{x^2}{2}\right)\Big|_0^1 = \boxed{\frac{\frac{y}{3}+\frac{1}{2}}{\frac{y}{2}+1} = E[X|Y]}$

2) Assume we have an experiment where the random variable $\mathbf{X}$ is assumed to follow an Erlang density, i.e.,

$$f_{\mathbf{X}}(x) \quad = \quad \theta^2 x e^{-\theta x} \quad 0 < x < \infty$$

Assume we preform the experiment n times with outcomes $x_1, x_2, \ldots, x_n$, What is the maximum liklihood estimate of $\hat{\theta}$ based on these observations?

$$L = \prod_{i=1}^{n} f_{\mathbf{X}_i}(x_i) = \theta^{2n} \left(\prod_{i=1}^{n} x_i \right) e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\ln(L) = 2n \ln(\theta) + \ln\left(\prod_{i=1}^{n} x_i \right) - \theta \sum_{i=1}^{n} x_i$$

$$\frac{\partial \ln(L)}{\partial \theta} = \frac{2n}{\theta} + 0 - \sum_{i=1}^{n} x_i = 0$$

$$\boxed{\hat{\theta} = \frac{2n}{\sum_{i=1}^{n} x_i}}$$

3) Assume $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are random vectors, not necessarily of the same size. Assume also that $K_{\mathbf{XX}}$, $K_{\mathbf{WW}}$ and $K_{\mathbf{WX}}$ are known. Now we make a new random vector

$$\underline{\mathbf{Y}} = A\underline{\mathbf{X}} + B\underline{\mathbf{W}} + \underline{C}$$

where A and B are constant matrices (not necessarily of the same size), and $\underline{C}$ is a constant vector. Determine an expression for $K_{\mathbf{YY}}$ in terms of these known quantities **ONLY**. Do **not** assume the means are zero.

Hint: $(FG)^T = G^T F^T$

$$K_{YY} = E\left\{[\underline{Y} - \underline{\mu}_Y][\underline{Y} - \underline{\mu}_Y]^T\right\} \qquad \underline{\mu}_Y = A\underline{\mu}_X + B\underline{\mu}_W + \underline{C}$$

$$\underline{Y} - \underline{\mu}_Y = (A\underline{X} + B\underline{W} + \underline{C}) - (A\underline{\mu}_X + B\underline{\mu}_W + \underline{C})$$

$$= A(\underline{X} - \underline{\mu}_X) + B(\underline{W} - \underline{\mu}_W)$$

$$K_{YY} = E\left\{\left[A(\underline{X} - \underline{\mu}_X) + B(\underline{W} - \underline{\mu}_W)\right]\left[A(\underline{X} - \underline{\mu}_X) + B(\underline{W} - \underline{\mu}_W)\right]^T\right\}$$

$$= E\left\{\left[A(\underline{X} - \underline{\mu}_X) + B(\underline{W} - \underline{\mu}_W)\right]\left[(\underline{X} - \underline{\mu}_X)^T A^T + (\underline{W} - \underline{\mu}_W)^T B^T\right]\right\}$$

$$= A\,E\left[(\underline{X} - \underline{\mu}_X)(\underline{X} - \underline{\mu}_X)^T\right]A^T + A\,E\left[(\underline{X} - \underline{\mu}_X)(\underline{W} - \underline{\mu}_W)^T\right]B^T$$

$$+ B\,E\left[(\underline{W} - \underline{\mu}_W)(\underline{X} - \underline{\mu}_X)^T\right]A^T + B\,E\left[(\underline{W} - \underline{\mu}_W)(\underline{W} - \underline{\mu}_W)^T\right]B^T$$

$$= \boxed{A K_{XX} A^T + A K_{WX}^T B^T + B K_{WX} A^T + B K_{WW} B^T = K_{YY}}$$

The two formulas may (or may not) be useful in the following problem.

The general formula for a multidimensional Gaussian density is

$$f_{\underline{\mathbf{X}}}(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\left[det(K_{\mathbf{XX}})^{\frac{1}{2}}\right]}\ exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T K_{\mathbf{XX}}^{-1}(\underline{x}-\underline{\mu})\right\}$$

The inverse of a 2 x 2 matrix is given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4) Assume the random vector $\underline{\mathbf{X}} = [\mathbf{X}_1\ \mathbf{X}_2]^T$ has the Gaussian density given by

$$f_{\underline{\mathbf{X}}}(\underline{x}) = \frac{1}{\pi\sqrt{3}}\ exp\left\{-\frac{1}{3}\left[2x_1^2 + 2x_1(x_2-1) + 2(x_2-1)^2\right]\right\}$$

Determine $\underline{\mu}$ and $K_{\mathbf{XX}}$

$\boxed{\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ by inspection}}$

$$\frac{1}{2}(\underline{x}-\mu_x)^T A(\underline{x}-\mu_x) = \frac{1}{2}\left[x_1\ (x_2-1)\right]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2-1 \end{bmatrix}$$

$$= \frac{1}{2}\left[x_1\ (x_2-1)\right]\begin{bmatrix} ax_1 + b(x_2-1) \\ bx_1 + c(x_2-1) \end{bmatrix} = \frac{1}{2}\left[ax_1^2 + bx_1(x_2-1) + bx_1(x_2-1) + c(x_2-1)^2\right]$$

$$\frac{a}{2}x_1^2 + bx_1(x_2-1) + \frac{c}{2}(x_2-1)^2 = +\frac{2}{3}x_1^2 + \frac{2}{3}x_1(x_2-1) + \frac{2}{3}(x_2-1)^2$$

$\frac{a}{2} = \frac{2}{3}$ $\quad a = 4/3$

$b = 2/3$ $\quad \frac{c}{2} = 2/3$ $\quad c = 4/3$

$$A = \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \qquad K_{xx} = A^{-1} = \frac{1}{\frac{16}{9}}\begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} = K_{xx}}$$

5) Assume $\underline{\mathbf{X}}$ is a zero mean random Gaussian vector with

$$K_{\mathbf{XX}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Determine a transformation matrix A and vector $\underline{b}$ such that

$$\underline{\mathbf{Y}} = A\underline{\mathbf{X}} + \underline{b}$$

and

$$K_{\mathbf{YY}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \underline{\mu}_{\mathbf{Y}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$K_{xx} - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \quad \det(K_{xx} - \lambda I) = (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda-2)(\lambda-4) = 0$$

$$\lambda_1 = 2 \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad \begin{matrix} a=-1 \\ b=1 \end{matrix} \quad \underline{\phi}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}} \qquad u = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_2 = 4 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad \begin{matrix} a=1 \\ b=1 \end{matrix} \quad \underline{\phi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$A = \Lambda_{xx}^{-\frac{1}{2}} u^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}} = \boxed{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} = A}$$