ECE-597: Probability, Random Processes, and Estimation
Computer Assignment #2

Due: Friday, March 27, 2015

You will use the results of this computer project in a future computer assignment.

In this project we will simulate a discrete-time Gaussian *random sequence*. We'll start with the relationship you derived in your last homework for a joint Gaussain pdf between random variables $\mathbf{X}$ and $\mathbf{Y}$, where

$$\begin{aligned} E(\mathbf{X}|\mathbf{Y}) &= \mu + \rho(\mathbf{Y} - \mu) \\ VAR(\mathbf{X}|\mathbf{Y}) &= \sigma^2(1-\rho^2) \end{aligned}$$

assuming $\sigma_X = \sigma_Y = \sigma$, and $\mu_x = \mu_y = \mu$.

We are now going to generate a sequence of jointly Gaussian random variables, with common mean μ and common variance σ^2. Specifically, we want $\mathbf{X}[1]$ and $\mathbf{X}[2]$ to be jointly Gaussian with mean μ, variance σ^2, and correlation coefficient ρ. We then want $\mathbf{X}[2]$ and $\mathbf{X}[3]$ to be jointly Gaussian with mean μ, variance σ^2, and correlation coefficient ρ. Similarly for $\mathbf{X}[3]$ and $\mathbf{X}[4]$, $\mathbf{X}[4]$ and $\mathbf{X}[5]$, etc. Eventually we are generating a sequence of Gaussian random variables, or a *discrete-time random sequence*. In order to do this, we take our formula above for the jointly Gaussian pair of random variables $\mathbf{X}$ and $\mathbf{Y}$ and index them so that they are adjacent pairs of random variables in the sequence. Specifically, let $\mathbf{X} \rightarrow \mathbf{X}[n+1]$ and $\mathbf{Y} \rightarrow \mathbf{X}[n]$. We will generate our random sequence using the formula

$$\mathbf{X}[n+1] = \mu + \rho(\mathbf{X}[n] - \mu) + \mathbf{W}[n]$$

We will assume that the random variable $\mathbf{W}[n]$ is an **independent** Gaussian random variable with zero mean and variance to be determined. **Note: We are using the important property that the sum of two Gaussian random variables is also a Gaussian random variable!**

1) Show that, if we assume that the random variable $\mathbf{X}[n]$ has mean μ, that random variable $\mathbf{X}[n+1]$ will have mean μ.

2) Show that, if we measure $\mathbf{X}[n]$, so that it is now a constant, that the conditional expectation of $\mathbf{X}[n+1]$ given $\mathbf{X}[n]$ is

$$E(\mathbf{X}[n+1]|\mathbf{X}[n]) = \mu + \rho(\mathbf{X}[n] - \mu)$$

3) We now need to determine the variance of $\mathbf{W}[n]$. Show that in order for

$$VAR(\mathbf{X}[n+1]|\mathbf{X}[n]) = \sigma^2(1-\rho^2)$$

that we need the variance of $\mathbf{W}[n]$ to be $\sigma_W^2 = \sigma^2(1-\rho^2)$. (*Hint: This uses problem 2 and is very easy.*)

4) You are to write a Matlab routine that will generate a random sequence with the conditional mean and variance given above. Your routine should have four inputs: (1) **Npoints** = the number of sample points to generate, (2) **rho** = the correlation coefficient between samples, (3) **sigma** = the standard deviation for the samples, and (4) **mu** = the mean for the samples. You are to generate the data using the formula

$$\mathbf{X}[n+1] \quad = \quad \mu + \rho(\mathbf{X}[n] - \mu) + \mathbf{W}[n]$$

To start the sequence, you should generate the first $\mathbf{X}[n]$ as a Gaussian random variable with mean μ and variance σ^2. From then on, for each new point of the random sequence ($\mathbf{X}[n+1]$), use the previously computed value of the sequence ($\mathbf{X}[n]$) and generate a new random Gaussian variable ($\mathbf{W}[n]$) with mean zero and variance $\sigma^2(1-\rho^2)$. Then $\mathbf{X}[n+1] \rightarrow \mathbf{X}[n]$ and generate the next point.

5) You should make another m-file (or function) which has as its input the random sequence you generated, and has as outputs the estimates of the mean, standard deviation, and correlation coefficient between samples. To estimate the mean and standard deviations of the sequence you can use the **mean** and **std** commands in Matlab. In order to estimate the correlation between pair of adjacent random variables in the random process, we will first use the following formula

$$\hat{R} \quad = \quad \frac{1}{N-1}\sum_{2}^{N} \mathbf{X}[i]\mathbf{X}[i-1]$$

(This is very similar to the way we would estimate the mean.) Finally, to estimate the correlation coefficient, use

$$\hat{\rho} \quad = \quad \frac{\hat{R} - \hat{\mu}^2}{\hat{\sigma}^2}$$

were the 'hats' mean the estimated values. Your estimates should be fairly close to the data you put into your program, but the mean and standard deviations will not work very well for large values of the correlation coefficient. Note that in these estimates we are assuming the process is **ergodic** (which we will talk about later).

6) Finally, you are to make some pretty plots of your results. First of all, use the subplot command to plot four generated random sequences on the same page. You should type '**orient landscape**' in Matlab before plotting to use more of the page.

7) You should use **Npoints** = 1000 for all plots. You should have one page for $\mu = 0$ and $\sigma = 1$, one page for $\mu = 2$ and $\sigma = 3$, and one page for $\mu = -1$ and $\sigma = 2$. For each mean and standard deviation, generate graphs using the correlation coefficients 0.01, 0.5, 0.9, and 0.99 as shown on the last page of this homework. Your pages should look like the one attached. You should note that if adjacent samples are highly correlated, the graph does not appear as "spikey". In engineering speak, the frequency content will be lower the higher the correlation.