ECE-597: Optimal Control
Homework #8

Due: Last day of class, 2007

1. *From Bryson* Another common form of the quadratic performance index, which we have already seen, is

$$J = \frac{1}{2}e_f^T Q_f e_f + \frac{1}{2}\int_{t_0}^{t_f} \left\{ y(t)^T Q_y y(t) + u(t)^T R_y u(t) \right\} dt$$

where $y(t) = Cx(t) + Du(t)$. Show that

$$\begin{aligned} Q &= C^T Q_y C \\ N &= C^T Q_y D \\ R &= R_y + D^T Q_y D \end{aligned}$$

2. Given the first order system with quadratic criterion

$$\begin{aligned} \dot{x}(t) &= u(t) \\ J &= \frac{1}{2} q \left\{ x(t_f) \right\}^2 + \frac{1}{2}\int_0^{t_f} u(t)^2 dt \end{aligned}$$

Use the equations

$$\begin{aligned} \dot{M} &= -MA \\ \frac{d}{dt}[\mathcal{Q}(t)]^{-1} &= = -MBR^{-1}B^T M^T \end{aligned}$$

show that

$$u(t) = \frac{-1}{\frac{1}{q} + T} x(t)$$

where $T = t_f - t$ is the time to go.

3. *From Bryson* Given the first order system with quadratic criterion

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) \\ J &= \frac{1}{2} S_f x_f^2 + \frac{1}{2}\int_0^{t_f} u(t)^2 dt \end{aligned}$$

where $x(0) = x_0$ and $x_f = x(t_f)$ and all quantities are scalars. Show that the closed loop control law is given by

$$u(t) = -K(t)x(t)$$

where $T = t_f - t =$ time to go, and

$$K(t) = \frac{e^{-2T}}{\frac{1}{S_f} + \frac{1}{2}\left(1 - e^{-2T}\right)}$$

You need to derive the solution in two different ways, using the equation

$$\dot{P} = AP + PA^T - BR^{-1}B^T$$

and then the equations

$$\begin{aligned} \dot{M} &= -MA \\ \frac{d}{dt}[\mathcal{Q}(t)]^{-1} &= = -MBR^{-1}B^T M^T \end{aligned}$$

4. Consider the tracking problem

$$\begin{aligned} \text{minimize } J &= \frac{1}{2}(y(T) - r(T))^T P(y(T) - r(T)) + \frac{1}{2}\int_{t_0}^{T} \left[(y-r)^T Q(y-r) + u^T Ru\right] dt \\ \text{subject to } & \dot{x} = Ax + Bu \\ & y(t) = Cx(t) \\ & P \geq 0,\ Q \geq 0,\ R > 0,\ P = P^T,\ Q = Q^T,\ R = R^T \end{aligned}$$

Here $r(t)$ is a reference signal we would like to track. Show that the solution to this problem can be written as:

$$\begin{aligned} \dot{S} &= -SA - A^T S + SBR^{-1}B^T S - C^T QC, \quad S(T) = C^T PC \\ \dot{g} &= -(A - BK)^T g + C^T Qr, \quad g(T) = -C^T Pr(T) \\ K(t) &= R^{-1}B^T S \\ u(t) &= -Kx - R^{-1}B^T g \end{aligned}$$