<u>ECE-521 Control Systems II</u>

Homework 5

Due at the beginning of class, Tuesday January 18 , 2005

1) Problem B-12-19 from the textbook. You should write out the equations to be solved, and then use Maple to final all possible solutions. Determine the eigenvalues for each possible P matrix. (*Ans*. 3 possible P matrices, $u(t) = -x_1 - x_2$).

2) Problem B-12-20 from the textbook. Do not use Maple or Matlab except to check yourself. (*Ans.* $u(t) = -x_1 - x_2\sqrt{\mu + 2}$)

<u>Preparation for Lab 4 (To be done individually, no Maple):</u>

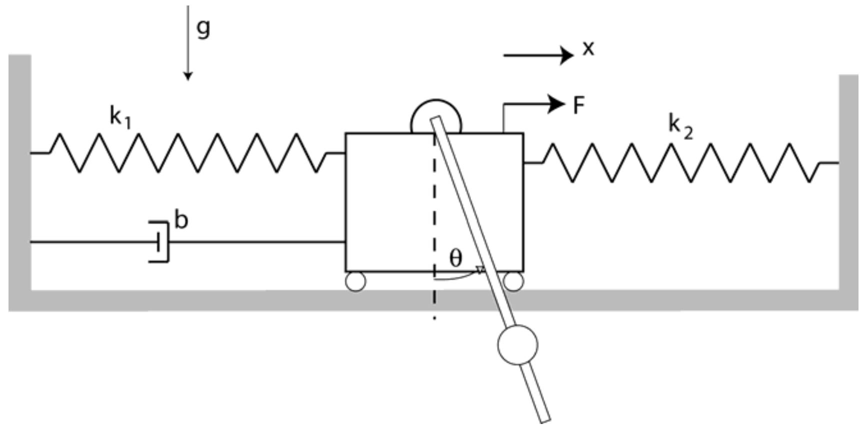
In this derivation we will make a state variable model for a regular pendulum (a pendulum hanging down) attached to the first cart. It will be easier to measure the parameters for a regular pendulum since it is a stable system. In the lab we will initially try and control the regular pendulum. Once this is working, we will try to control an inverted pendulum. To go from the model of a regular pendulum to the model of an inverted pendulum we use the substitution $l \to -l$.

3) In a previous homework we derived the equations of motion for an inverted pendulum on a cart as

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos(\theta) + ml\dot{\theta}^2\sin(\theta) + c\dot{x} + kx = F$$
$$(J+ml^2)\ddot{\theta} - ml\ddot{x}\cos(\theta) - mgl\sin(\theta) = 0$$

The mass of the cart is M , the mass at the center of mass of the pendulum is m, the moment of inertial of the pendulum about its center of mass is J, L is the length of the pendulum, and l is the distance from the pivot to the center of mass of the pendulum. The angle θ is measured counterclockwise from straight up, x is the displacement of the first cart (positive to the right), and g is the gravitational constant.

In order to derive the model parameters we need, we will first model a regular pendulum configuration. This will be easier to deal with since it is an inherently stable system. In order to change our model from the inverted pendulum to a regular pendulum, we make the transformation $l \to -l$ in the above equations. Now the angle θ is measured counterclockwise from straight down. The configuration for the regular pendulum is shown below:



a) Show that the equations of motion for the regular pendulum can be written as

$$\begin{aligned}(J+ml^2)\ddot{\theta}+ml\cos(\theta)\ddot{x}+mgl\sin(\theta) &= 0\\ \left(M+m\right)\ddot{x}+ml\ddot{\theta}\cos(\theta)-ml\dot{\theta}^2\sin(\theta)+c\dot{x}+kx &= F\end{aligned}$$

b) Using the small angle/small velocity assumption, show that we can approximate the above equations of motion as

$$\begin{aligned}(J+ml^2)\ddot{\theta}+ml\ddot{x}+mgl\theta &\approx 0\\ \left(M+m\right)\ddot{x}+ml\ddot{\theta}+c\dot{x}+kx &\approx F\end{aligned}$$

c) We can rewrite the first equation above as

$$\frac{1}{\omega_\theta^2}\ddot{\theta}+\frac{1}{g}\ddot{x}+\theta=0$$

What is ω_θ^2?

d) If we assume the cart is fixed, then $\ddot{x}=0$ and we have

$$\ddot{\theta}+\omega_\theta^2\theta=0$$

This is the equation for a simple pendulum. If the pendulum is deflected a small angle and released, it will oscillate with frequency ω_θ. If we measure the period of the oscillations T_θ how do we find ω_θ?

e) We can rewrite the second equation from step (b) as

$$\frac{1}{\omega_1^2}\ddot{x}+\frac{2\zeta_1}{\omega_1}x+x+K_1\ddot{\theta}=K_2F$$

Find expressions for ω_1, ζ_1, K_1, and K_2 in terms of m, M, k, and l. If we assume there is not input ($F=0$) and the pendulum does not move very much ($\ddot{\theta}\approx 0$) then we can use the log-decrement method to get initial estimates of ω_1 and ζ_1.

f) Assuming we apply a step input of amplitude A to the cart, show that in steady state we get

$$K_2=\frac{x_{ss}}{A}$$

g) Show that

$$\frac{\Theta(s)}{X(s)}=-\frac{\omega_\theta^2}{g}\left(\frac{s^2}{s^2+\omega_\theta^2}\right)$$

We need to measure the gravitational constant in cm, since all other distances are measured in cm.

h) Show that

$$\frac{X(s)}{F(s)}=\frac{\omega_1^2K_2(s^2+\omega_\theta^2)}{\left(1-K_1\omega_1^2\frac{\omega_\theta^2}{g}\right)s^4+\left(2\zeta_1\omega_1\right)s^3+(\omega_1^2+\omega_\theta^2)s^2+(2\zeta_1\omega_1\omega_\theta^2)s+\omega_1^2\omega_\theta^2}$$

We can use this expression to determine K_1 and get better estimates of ω_1 and ζ_1.

i) We can rewrite our linearized dynamical equations as

$$\begin{aligned}\ddot{\theta} &\approx -\frac{1}{g}\omega_\theta^2\ddot{x}-\omega_\theta^2\theta\\ \ddot{x} &\approx -2\zeta_1\omega_1\dot{x}-\omega_1^2x-K_1\omega_1^2\ddot{\theta}+K_2\omega_1^2F\end{aligned}$$

By substituting the second equation into the first equation, show that we get

$$\ddot{\theta}\approx\frac{1}{\Delta}\left(\frac{\omega_1^2\omega_\theta^2}{g}\right)x+\frac{1}{\Delta}\left(\frac{2\zeta_1\omega_1\omega_\theta^2}{g}\right)\dot{x}+\frac{1}{\Delta}\left(-\omega_\theta^2\right)\theta+\frac{1}{\Delta}\left(-\frac{\omega_1^2\omega_\theta^2K_2}{g}\right)F$$

where

$$\Delta=1-\left(\frac{K_1\omega_1^2\omega_\theta^2}{g}\right)$$

and substituting the first equation into the second equation we get

$$\ddot{x} \approx \frac{1}{\Delta}\left(-\omega_1^2\right)x + \frac{1}{\Delta}\left(-2\zeta_1\omega_1\right)\dot{x} + \frac{1}{\Delta}\left(K_1\omega_1^2\omega_\theta^2\right)\theta + \frac{1}{\Delta}\left(K_2\omega_1^2\right)F$$

j) Defining $q_1 = x$, $q_2 = \dot{x}$, $q_3 = \theta$, and $q_4 = \dot{\theta}$, show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{\omega_1^2}{\Delta}\right) & -\left(\frac{2\zeta_1\omega_1}{\Delta}\right) & \left(\frac{K_1\omega_1^2\omega_\theta^2}{\Delta}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{\omega_1^2\omega_\theta^2}{g\Delta}\right) & \left(\frac{2\zeta_1\omega_1\omega_\theta^2}{g\Delta}\right) & -\left(\frac{\omega_\theta^2}{\Delta}\right) & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{K_2\omega_1^2}{\Delta}\right) \\ 0 \\ -\left(\frac{\omega_1^2\omega_\theta^2 K_2}{g\Delta}\right) \end{bmatrix} F$$

k) If we now want to model the inverted pendulum ($l \to -l$ to go back to the inverted pendulum), which terms change in the matrices above?

l) When we try and fit the frequency response data we see in the lab we will often get an unusual response. To understand this response we will analytically try and show what is happening. If you've not screwed up, you should have obtained values of

$$\begin{aligned} K_1 &= \frac{ml}{k} \\ \omega_1^2 &= \frac{k}{M+m} \\ \omega_\theta^2 &= \frac{mgl}{J+ml^2} \end{aligned}$$

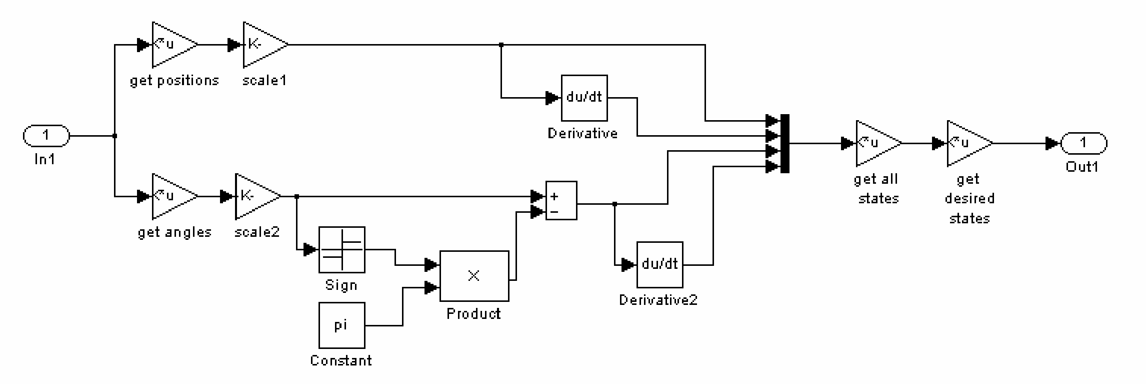
If we assume that the mass of the cart and pendulum attachment is much larger than the mass at the center of mass of the pendulum, then we have $M \gg m$. Secondly, J is the moment of inertia about the center of mass of the pendulum, m is the mass at the center of mass of the pendulum, and l is the distance from the pivot to the center of mass of the pendulum. For our systems, the pendulum bars have negligible mass and all of the mass is essentially concentrated at the center of mass. Hence we have $ml^2 \gg J$. Using these two assumptions, show that $\Delta \approx 1$.

m) Assuming $\Delta \approx 1$, show that

$$\frac{X(s)}{F(s)} \approx \frac{\omega_1^2 K_2}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2}$$

That is, there is a pole/zero cancellation! This is the effect you will often see in lab.

n) In trying to control the inverted pendulums on the ECP systems, there is one last little problem we need to deal with. The ECP system sets zero degrees to be where ever the pendulum is when the system is reset. For the regular pendulum, in which the pendulum is hanging down, there is no problem since we just reset the system when the pendulum is at rest pointing down. However, for the inverted pendulum, we have a problem in that we really won't be able to hold the pendulum vertical when resetting the system. Even slight deviations from exactly vertical will screw us up. To solve this, we will modify how the Simulink model of the ECP system identifies angles by using the following piece of Simulink code:



This is from the **manipulate data** part of **ECP_Model210_Inverted_Pendulum_Template.mdl** you will be using in lab. The lower path usually gets the angle of the pendulum (in radians) and then takes its derivative. However, there have been a few changes made so our Simulink model believes that vertical (up) is really 0 degrees. For this problem, the ECP system believes 0 degrees is straight down, and assume the angle coming out of the scale2 block is 90 degrees plus or minus 5 degrees (nearly straight up). What do these angles map to (what is the output just after the summation?)

4) Modify (save under a new filename before modifying) the two degree of freedom Simulink model from homework 2 and corresponding Matlab code to work with the ***regular*** pendulum model. (The state model is available one the course website.) Specifically, you need to

- Modify the matrix *get_desired_states* so that when you do the lab, the ECP system will output states x1, x1_dot, theta, and theta_dot
- Have 4 model outputs, m_x1, m_x1_dot, m_theta, m_theta_dot
- Plot the position of the cart, the velocity of the cart, the position of the pendulum, and the velocity of the pendulum. All plots should be neatly organized on one page.
- Set the input of the system to zero (this is a *regulator problem*, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.015 radians and all other initial conditions to zero.
- Utilize the linear quadratic regulator or pole placement method to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight down and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4 and the system should come to steady state in less than 0.1 seconds.

You will need to turn in you plot, your Simulink code, and your Matlab code.

5) Utilize the results of problem 6 to model the ***inverted*** pendulum. The only thing you should need to change is the state model. (The state model is available one the course website.) Specifically, you need to

- Set the input of the system to zero (this is a *regulator problem*, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.015 radians and all other initial conditions to zero.
- Utilize the linear quadratic regulator or pole placement method to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight up and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4. Limiting the cart motion is usually the most difficult part.

You will need to turn in you plot for this part..