<u>ECE-521 Control Systems II</u>

Homework 2

Due at the beginning of class, Tuesday December 14, 2004

1) Find the eigenvalues and corresponding eigenvectors for the following matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

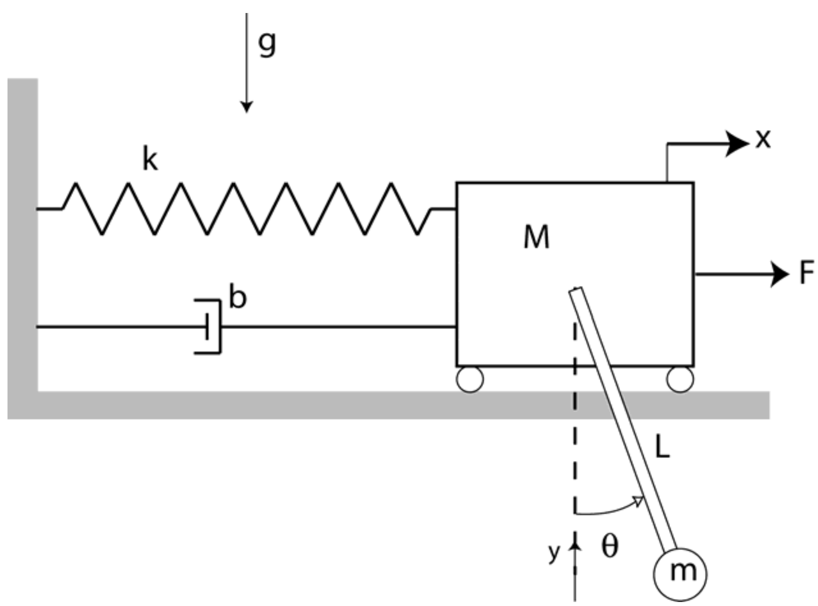
2) Find the eigenvalues and corresponding eigenvectors for the following matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

3) Find the eigenvalues and corresponding eigenvectors for the following matrix. How many linearly independent eigenvectors can you find?

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

4) For the following pendulum system (assume the pendulum arm is massless)

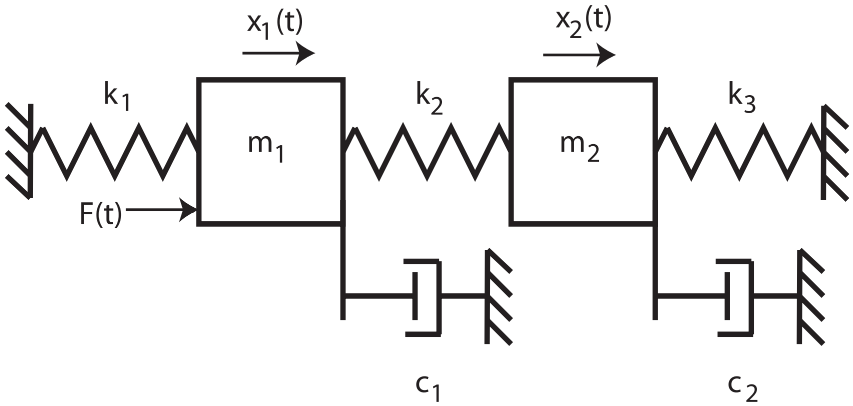


show that the equations of motion are given by

$$(M+m)\ddot{x} + mL\cos(\theta)\ddot{\theta} - mL\sin(\theta)\dot{\theta}^2 + b\dot{x} + kx = F$$
$$mL^2\ddot{\theta} + mL\cos(\theta)\ddot{x} + mgL\sin(\theta) = 0$$

<u>Preparation for Lab 2 (to be done individually, No Maple):</u>

5) Consider the following model of the two degree of freedom system we will be using in lab 2.



a) Using Lagrangian dyanamics, show that the equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 &= F + k_2x_2 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 &= k_2x_1 \end{aligned}$$

b) Defining $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, and $q_4 = \dot{x}_2$, show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & -\left(\frac{c_1}{m_1}\right) & \left(\frac{k_2}{m_1}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{k_2}{m_2}\right) & 0 & -\left(\frac{k_2+k_3}{m_2}\right) & -\left(\frac{c_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{m_1}\right) \\ 0 \\ 0 \end{bmatrix} F$$

In order to get the A and B matrices for the state variable model, we need to determine all of the quantities in the above matrices. The C matrix will be determined by what we want the output of the system to be.

c) If we want the output to be the position of the first cart, what should C be? If we want the output to be the position of the second cart what should C be?

d) Now we will rewrite the equations from part (a) as

$$
\begin{aligned}
\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 &= \frac{k_2}{m_1}x_2 + \frac{1}{m_1}F \\
\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 &= \frac{k_2}{m_2}x_1
\end{aligned}
$$

We will get our initial estimates of ζ_1, ω_1, ζ_2, and ω_2 using the log-decrement method (assuming only one cart is free to move at a time). Assuming we have measured these parameters, show how $A_{2,1}$, $A_{2,2}$, $A_{4,3}$, and $A_{4,4}$ can be determined.

e) By taking the Laplace transforms of the equations from part (d), show that we get the following transfer function

$$
\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) - \frac{k_2^2}{m_1 m_2}}
$$

f) It is more convenient to write this as

$$
\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)}
$$

By equating powers of s in the denominator of the transfer function from part (e) and this expression you should be able to write down four equations. The equations corresponding to the coefficients of s^3, s^2, and s do not seem to give us any new information, but they will be used to get consistent estimates of ζ_1 and ω_1. The equation for the coefficient of s^0 will give us a new relationship for $\frac{k_2^2}{m_1 m_2}$ in terms of the parameters we will be measuring.

g) We will actually be fitting the frequency response data to the following transfer function

$$
\frac{X_2(s)}{F(s)} = \frac{K_2}{\left(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1\right)\left(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1\right)}
$$

What is K_2 in terms of the parameters of part (f)?

h) Using the transfer function in (f) and the Laplace transform of the second equation in part (d), show that the transfer function between the input and the position of the first cart is given as

$$\frac{X_1(s)}{F(s)}=\frac{\frac{1}{m_1}\left(s^2+2\zeta_2\omega_2 s+\omega_2^2\right)}{(s^2+2\zeta_a\omega_a s+\omega_a^2)(s^2+2\zeta_b\omega_b s+\omega_b^2)}$$

i) This equation is more convenient to write in the form

$$\frac{X_1(s)}{F(s)}=\frac{K_1\left(\frac{1}{\omega_2^2}s^2+\frac{2\zeta_2}{\omega_2}s+1\right)}{(\frac{1}{\omega_a^2}s^2+\frac{2\zeta_a}{\omega_a}s+1)(\frac{1}{\omega_b^2}s^2+\frac{2\zeta_b}{\omega_b}s+1)}$$

What is K_1 in terms of the quantities given in part (h)?

j) Verify that $A_{4,1}=\frac{k_2}{m_2}=\frac{K_2}{K_1}\omega_2^2$

k) Verify that $A_{2,3}=\frac{k_2}{m_1}=\frac{\omega_1^2\omega_2^2-\omega_a^2\omega_b^2}{A_{4,1}}$

l) Verify that $B_2=\frac{1}{m_1}=\frac{K_2\omega_a^2\omega_b^2}{A_{4,1}}$. Note that this term contains all of the scaling and unit conversions.

6) Modify (save under a new filename before modifying) the one degree of freedom Simulink model (*Basic_1dof_State_Variable_Model.mdl*) and corresponding Matlab code to work with a two degree of freedom model. (The state model is available one the course website.) Specifically, you need to

- Modify the matrix *get_desired_states* so that when you do the lab, the ECP system will output states x1, x1_dot, x2, and x2_dot.
- Have 4 model outputs, m_x1, m_x1_dot, m_x2, m_x2_dot
- In addition to the plots for the one degree of freedom model, plot the position and velocity of the second cart. All plots should be neatly organized on one page
- Using the state variable model on the web page, place the four closed loop poles at -10, -15, -20+10j, -20-10j
- By changing the C vector, control the position of the first cart so it follows a 1 cm step input.
- By changing the C vector, control the position of the second cart so it follows a 1 cm step input.

You will need to turn in two plots, your Simulink code, and your Matlab code.

7) Utilizing the Simulink code from part 6, utilize the linear quadratic regulator method for choosing the state feedback gain K so that first cart follows a step input of amplitude 1 cm (we are trying to control the position of the first cart so that should be the system output). The control effort must remain below 0.4 and the settling time must be less than 0.3 seconds.

The linear quadratic regulator finds the gain K to minimize

$$J = \int_0^\infty \left[x^T(t) Q x(t) + u(t)^T R u(t) \right] dt$$

where

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ u(t) &= -Kx(t) \end{aligned}$$

For this system, Q is a 4x4 positive definite matrix, and R is a scalar. Since we will use a diagonal matrix for Q and for our system $u(t)$ is a scalar, we can rewrite J as

$$J = \int_0^\infty \left[q_1 x_1^2(t) + q_2 \dot{x}_1^2(t) + q_3 x_2^2(t) + q_4 \dot{x}_2^2 + Ru^2(t) \right] dt$$

A large value of R penalizes a large control signal, a large value of q_1 will penalize the position of the first cart, a large value of q_2 will penalize a large value of the velocity of the first cart, a large value of q_3 will penalize the position of the second cart, while a large value of q_4 penalizes the velocity of the second cart. All of the q_i should be zero or positive.

It's easiest to find K using the following command in Matlab: $K = lqr(A, B, diag([q_1 \quad q_2 \quad q_3 \quad q_4], R);$

You will have to try different values of the q_i to find an acceptable controller. Later in the course we will see how Matlab determines these gains. You need to turn in one plot, as well as your Matlab code.