ECE 521: Control Systems II

Homework #2

Due Tuesday March 23

1) With $\underline{u}_o^T = [0, -1.5]$ show that the equilibrium point of the following systems is $\underline{x}_o^T = [1, 0.5]$ (just plug these into the equations, don't try to derive them...)

$$\begin{aligned} \dot{x}_1 &= 4x_1 + 2x_2^2 + u_1 + 3u_2 \\ \dot{x}_2 &= x_1^3 + x_2 + 2u_1 + u_2 \end{aligned}$$

Linearize these equations about these nominal points and write the result in state variable form.

2) Suppose we want to minimize a function while satisfying a constraint. For example, find the point in the plane $x + y = 5$ nearest the origin.

$$\begin{array}{lcl} \text{minimize} & x^2 + y^2 & \text{(distance from origin)} \\ \text{subject to} & x + y - 5 = 0 & \text{(must lie in plane)} \end{array}$$

We do this with Lagrange multipliers (λ) and form the minimization problem

$$\text{minimize } L(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 5)$$

where x, y, and λ are now variables. Set $\frac{dL}{dx} = 0, \frac{\partial L}{dy} = 0$, and $\frac{\partial L}{d\lambda} = 0$ and show the optimal point is $x = 5/2, y = 5/2$.

3) We can also add constraints to vector minimization problems. Assume we want to find the minimum value of $\underline{x}^T A \underline{x}$ where A is a symmetric matrix, subject to the constraint the magnitude of the vector $\underline{x}$ is one. We can write this as

$$\begin{aligned} \text{minimize} \quad & f(\underline{x}) = \underline{x}^T A \underline{x} \\ \text{subject to} \quad & 1 - \underline{x}^T \underline{x} = 0 \end{aligned}$$

form

$$L(\underline{x}, \lambda) = f(\underline{x}) + \lambda(1 - \underline{x}^T \underline{x})$$

Now set $\frac{\partial L}{\partial \underline{x}} = 0$ and $\frac{\partial L}{\partial \lambda} = 0$ and show that the minimum value is given by the smallest eigenvalue of A.

4) Find $\underline{x}$ such that

$$A\underline{x} = \underline{y}$$

and $\underline{x}$ is the minimum norm solution.

Hint: the minimum norm solution is the $\underline{x}$ solution to the problem:

$$\begin{aligned} \text{minimize} \quad & \underline{x}^T \underline{x} \\ \text{subject to} \quad & A\underline{x} = \underline{y} \end{aligned}$$

In this problem the Lagrange multiplier is a vector, and

$$L(\underline{x}, \underline{\lambda}) = \underline{x}^T \underline{x} + \underline{\lambda}^T (A\underline{x} - \underline{y})$$

You should be able to show $\underline{x} = A^T (AA^T)^{-1} \underline{y}$

5) Consider the discrete time state variable system

$$\underline{x}_{k+1} = G\underline{x}_k + H\underline{u}_k$$

with the initial state $\underline{x}_0 = \underline{0}$.

a) Show that after 3 time steps (k=0,1,2), we have the system of equations

$$\underline{x}_3 = \begin{bmatrix} G^2H \; GH \; H \end{bmatrix} \begin{bmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}$$

b) Assume we want to go from the origin to the final state $\underline{x}_f$ in three time steps with a penalty on the amount of input. Show that we can formulate this as

$$\begin{array}{ll} \text{minimize} & \underline{u}^T R \underline{u} \\ \text{subject to} & \underline{x}_f - Q\underline{u} = 0 \end{array}$$

what are $\underline{u}$ and Q?

c) Assuming that Q and R have full rank and R is a diagonal matrix, show that the control $\underline{u}$ which minimizes the function is given by

$$\underline{u} = R^{-1}Q^T(QR^{-1}Q^T)^{-1}\underline{x}_f$$

(note: **Do Not** assume Q^{-1} exists!).
d) Now assume

$$G = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \underline{x}_0 = \underline{0}, \underline{x}_f = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and $R = diagonal(1,2,3,4,5,6)$, i.e. $r_{11} = 1, r_{22} = 2, ...r_{66} = 6$. What is the input $\underline{u}$ to minimize $\underline{u}^T R \underline{u}$ and take the system from 0 to $\underline{x}_f$?

6) Consider the following state variable system

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned}$$

a) Now consider transforming this to a new set of basis vectors (transforming the basis). Let $\underline{x} = Q\underline{z}$ where the columns of Q are the new basis vectors. Assuming that Q^{-1} exists, show that in the new basis the state variable system becomes:

$$\begin{aligned} \underline{\dot{z}} &= Q^{-1}AQ\underline{z} + Q^{-1}B\underline{u} \\ \underline{y} &= CQ\underline{z} + D\underline{u} \end{aligned}$$

b) Now consider the state system with the representation

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [1\ 0]\,\underline{x} \end{aligned}$$

Compute the eigenvalues and eigenvectors of the A matrix, call them $\underline{q}_1$ and $\underline{q}_2$.

c) Construct the matrix $Q = \left[\underline{q}_1 \underline{q}_2\right]$.

d) Rewrite the system using the eigenvectors as the basis vectors.

e) Determine an expression for A^2 in terms of A and I and then show explicitly that the matrix A satisfies its own characteristic equation by using the A matrix and evaluating both sides of the equation.

f) Determine an expression for e^{At} using the Cayley-Hamilton method (matching functions on eigenvalues).

7) For matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

a) Find the eigenvalues and characteristic equation for A.

b) Determine an expression for A^2 in terms of A and I and then show explicitly that the matrix A satisfies its own characteristic equation by using the A matrix and evaluating both sides of the equation.

c) Determine and expression for e^{At} using the Cayley-Hamilton method (matching functions on eigenvalues).

d) Determine e^{At} using the Laplace transform method.

8) Assume a matrix A is determined to have the following eigenvalues, $\lambda = -1, -1, -1, -2, -2, -3$. Determine the simultaneous equations that need to be solved to determine e^{At}. DO NOT SOLVE!

9) When we have distinct eigenvalues and a single input system, the state equations will be decoupled, and we can write

$$\dot{x}_i = \lambda_i x_i + \hat{b}_i u$$

Determine the transfer functions for both continuous and discrete time instances ($H_i(s) = X_i(s)/U(s)$) and the conditions on the system eigenvalues for stability.

10) For

$$A = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}$$

a) Determine $P(A)$ for $P(x) = x^4 - 5x^3 - x^2 + 6x + 1$ by computing the quotient $Q(x)$ and the remainder $R(x)$

$$P(x) = Q(x)\Delta(x) + R(x)$$

and using $P(A) = R(A)$

b) Compute $f(A) = e^{At}$ using the Cayley-Hamilton Theory method (matching function on eigenvalues).

1

$$\delta\dot{\underline{x}} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \delta\underline{x} + \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \delta u$$

5 $\underline{u}^T = [0.71 \ -0.52 \ -0.23 \ -0.04 \ 0.69 \ 0.59]$

6

$$e^{At} = \begin{bmatrix} 2e^{2t} - e^{3t} & e^{2t} - e^{3t} \\ 2e^{3t} - 2e^{2t} & 2e^{3t} - e^{2t} \end{bmatrix}$$

7

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix}$$

10 $P(A) = A + I$

$$e^{At} = \begin{bmatrix} 0.25e^{5t} + 0.75e^{t} & 0.433e^{5t} - 0.433e^{t} \\ 0.433e^{5t} - 0.433e^{t} & 0.75e^{5t} + 0.25e^{t} \end{bmatrix}$$