ECE-520: Linear Control Systems
Homework 2

Due: Thursday December 8 at 5 PM

Problems

1) From Ogata, B-2-3. (*Answer:* $Y(z)=\dfrac{T^2 z^{-1}e^{-aT}(1+z^{-1}e^{-aT})}{(1-z^{-1}e^{-aT})^3}$)

2) From Ogata, B-2-4. (*Answer:* $X(z)=\dfrac{z^{-2}+2}{(1-2z^{-1})^2(1-z^{-1})}$)

3) From Ogata, B-2-5 (You'll need the results from A-2-4. *Answer:* $X(z)=\dfrac{1}{1-z^{-1}}\dfrac{1}{1-az^{-1}}$)

4) From Ogata, B-2-6.

5) From Ogata, B-2-7. (Assume sampling every second, it's easier if you write $x(k)=u(k)-\delta(k)-\delta(k-1)\cdots$ *Answer:* $X(z)=\dfrac{1}{3}\dfrac{z^{-3}+z^{-4}+z^{-5}}{1-z^{-1}}$)

6) Using the fact that $e^a=\sum_{k=0}^{\infty}\dfrac{a^k}{k!}$, find the z transform of

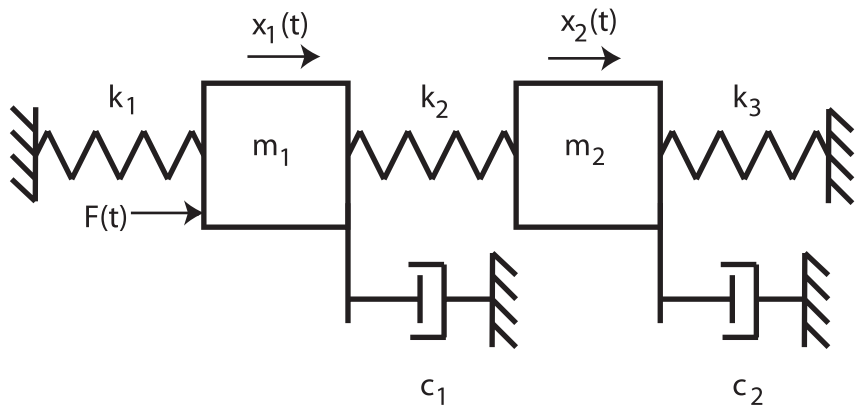
a) $f(k)=\dfrac{\gamma^k}{k!}u(k)$

b) $f(k)=\dfrac{(\ln(\gamma))^k}{k!}u(k)$

(*Answers:* $F(z)=e^{\frac{\gamma}{z}}, F(z)=\gamma^{\frac{1}{z}}$)

Preparation for Lab 2 (to be done individually, No Maple)

7) Consider the following model of the two degree of freedom system we will be using in lab 2.



a) Draw free body diagrams for each mass and show that the equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 &= F + k_2x_2 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 &= k_2x_1 \end{aligned}$$

b) Defining $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, and $q_4 = \dot{x}_2$, show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & -\left(\frac{c_1}{m_1}\right) & \left(\frac{k_2}{m_1}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{k_2}{m_2}\right) & 0 & -\left(\frac{k_2+k_3}{m_2}\right) & -\left(\frac{c_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{m_1}\right) \\ 0 \\ 0 \end{bmatrix} F$$

In order to get the A and B matrices for the state variable model, we need to determine all of the quantities in the above matrices. The C matrix will be determined by what we want the output of the system to be.

c) If we want the output to be the position of the first cart, what should C be? If we want the output to be the position of the second cart what should C be?

d) Now we will rewrite the equations from part (a) as

$$
\begin{aligned}
\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 &= \frac{k_2}{m_1}x_2 + \frac{1}{m_1}F \\
\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 &= \frac{k_2}{m_2}x_1
\end{aligned}
$$

We will get our initial estimates of ζ_1, ω_1, ζ_2, and ω_2 using the log-decrement method (assuming only one cart is free to move at a time). Assuming we have measured these parameters, show how $A_{2,1}$, $A_{2,2}$, $A_{4,3}$, and $A_{4,4}$ can be determined.

e) By taking the Laplace transforms of the equations from part (d), show that we get the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) - \frac{k_2^2}{m_1 m_2}}$$

f) It is more convenient to write this as

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)}$$

By equating powers of s in the denominator of the transfer function from part (e) and this expression you should be able to write down four equations. The equations corresponding to the coefficients of s^3, s^2, and s do not seem to give us any new information, but they will be used to get consistent estimates of ζ_1 and ω_1. The equation for the coefficient of s^0 will give us a new relationship for $\frac{k_2^2}{m_1 m_2}$ in terms of the parameters we will be measuring.

g) We will actually be fitting the frequency response data to the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{K_2}{\left(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1\right)\left(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1\right)}$$

What is K_2 in terms of the parameters of part (f)?

h) Using the transfer function in (f) and the Laplace transform of the second equation in part (d), show that the transfer function between the input and the position of the first cart is given as

$$\frac{X_1(s)}{F(s)} = \frac{\frac{1}{m_1}\left(s^2 + 2\zeta_2\omega_2 s + \omega_2^2\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)}$$

i) This equation is more convenient to write in the form

$$\frac{X_1(s)}{F(s)} = \frac{K_1\left(\frac{1}{\omega_2^2}s^2 + \frac{2\zeta_2}{\omega_2}s + 1\right)}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)}$$

What is K_1 in terms of the quantities given in part (h)?

j) Verify that $A_{4,1} = \frac{k_2}{m_2} = \frac{K_2}{K_1}\omega_2^2$

k) Verify that $A_{2,3} = \frac{k_2}{m_1} = \frac{\omega_1^2\omega_2^2 - \omega_a^2\omega_b^2}{A_{4,1}}$

l) Verify that $B_2 = \frac{1}{m_1} = \frac{K_2\omega_a^2\omega_b^2}{A_{4,1}}$. Note that this term contains all of the scaling and unit conversions.