ECE-520: Discrete-Time Control Systems
Homework 3

Due: Tuesday December17 in class

1) Prove or disprove the following claims: if $u, v,$ and w are linearly independent vectors, then so are

a) $u, u+v, u+v+w$
b) $u+2v-w, u-2v-w, 4v$
c) $u-v, v-w, w-u$
d) $-u+v+w, u-v+w, -u+v-w$

Note: You must do this for arbitrary vectors. ***Do Not*** assume $u, v,$ and w are specific vectors.

2) For the following matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

a) Find a set of vectors that form a basis for the null space of A?

b) Is the vector $\underline{n} = \begin{bmatrix} 2 & 2 & -2 & 2 \end{bmatrix}^T$ in the null space of A? That is, can you represent this vector as a linear combination of your basis vectors?

c) Is the vector $\underline{a} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ in the range (column) space of A?

3) For the following matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

a) Find a set of vectors that form a basis for the null space of A?

b) Is the vector $\underline{n} = \begin{bmatrix} 2 & 6 & -2 & -1 \end{bmatrix}^T$ in the null space of A? That is, can you represent this vector as a linear combination of your basis vectors?

c) Is the vector $\underline{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ in the range (column) space of A?

4) For the following matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 3 & 5 \end{bmatrix}$$

a) Find the rank of A (the number of linearly independent columns)

b) Determine two vectors that span the null space of A.

c) Determine two vectors that span the row space of A.

d) Show that any vector in the row space of A is orthogonal to any vector in the null space of A.

e) Determine two vectors that span the column space of A.

5) Suppose we want to minimize a function while satisfying a constraint. For example, find the point in the plane $x + y = 5$ closest to the origin. We want to write this as a minimization problem with a constraint, such as

minimize $x^2 + y^2$ (distance from origin)
subject to x+y-5 = 0 (constraint)

We do this with Lagrange multipliers (λ) and form the minimization problem

$$\text{minimize } L(x, y, \lambda) = x^2 + y^2 + \lambda(x + y - 5)$$

To solve the problem we now set $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$. Show that the optimal point is $x = y = \frac{5}{2}$.

6) Consider the discrete-time state variable system

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with initial state $\underline{x}(0) = \underline{0}$.

a) Show that after three steps $(k = 0,1,2)$ we have the system of equations

$$\underline{x}(3) = \begin{bmatrix} G^2H & GH & H \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

b) Assume we want to go from the origin to the final state $\underline{x}_f$ in three time steps with a penalty on the amount of input (i.e., the signal energy) We can formulate this problem as

$$\begin{array}{ll} \text{minimize} & \underline{u}^T R \underline{u} \quad \text{(minimize engergy)} \\ \text{subject to} & \underline{x}_f - Q\underline{u} \quad \text{(constraint: must reach final state)} \end{array}$$

R is a *symmetric* weighting matrix, indicating how much to penalize the input energy at each time step. What are Q and $\underline{u}$?

c) We can again solve this problem using Lagrange multipliers. The form of the Lagrange multiplier is chosen so the L function makes mathematical sense. For example, for this problem, the function to be minimized is a scalar, so the Lagrange multiplier must be chosen to make the problem a scalar problem. Specifically, we form

$$L(\underline{u},\underline{\lambda}) = \underline{u}^T R\underline{u} + \underline{\lambda}^T (Q\underline{u} - \underline{x}_f)$$

Assuming R^{-1} exists but Q^{-1} does not exist, show that the optimal control signal is

$$\underline{u} = R^{-1}Q^T (QR^{-1}Q^T)^{-1} \underline{x}_f$$

d) How would your answer change if $\underline{x}(0) \neq \underline{0}$?

e) For $G = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, determine the matrix Q and then determine vector(s) that span the null space of Q

f) For $\underline{x}_f = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$, find a control vector $\underline{u}$ that takes the system from the origin to $\underline{x}_f$ that minimizes the energy (is a minimum norm solution).

g) Show that your control vector $\underline{u}$ to part **e** is orthogonal to the vector(s) that span the null space of Q, hence $\underline{u}$ has no components in the null space of Q.