ECE-497: Inverse Problems in Engineering

Homework #6

Due: Friday January 30, 2003

Motivation

In this homework, we examine the use of Generalized Cross Validation (GCV) for choosing the regularization parameter.

1) In this problem, we will determine how to use GCV for the case of higher order Tikhonov regularization. GCV computes the b that minimizes the function

$$V(b) \;=\; \frac{||\underline{x} - \underline{d}||^2}{\text{Trace}(I - A(b))^2}$$

From the last homework, we know the $\underline{g}$ that is the solution to the regularization problem

$$\begin{aligned} \min \Pi &= ||\underline{x} - \underline{d}||^2 + b||\mathbf{R}\underline{g}||^2 \\ \text{subject to } \underline{x} &= \mathbf{Q}\underline{g} \end{aligned}$$

is given by

$$\underline{g} \;=\; \left(\mathbf{Q}^T\mathbf{Q} + b\mathbf{R}^T\mathbf{R}\right)^{-1}\mathbf{Q}^T\underline{d}$$

a) Using the generalized singular value decomposition (gsvd), we can write

$$\begin{aligned} \mathbf{Q} &= \mathbf{UCX}^T \\ \mathbf{R} &= \mathbf{VSX}^T \end{aligned}$$

where $\mathbf{U}$ and $\mathbf{V}$ are *unitary* matrices, $\mathbf{C}$ and $\mathbf{S}$ are *diagonal* matrices with elements s_i and c_i on the diagonals, and

$$\mathbf{C}^T\mathbf{C} + \mathbf{S}^T\mathbf{S} = \mathbf{I}$$

Assuming $\mathbf{X}^{-1}$ exists, and defining

$$\underline{\tilde{d}} \;=\; \mathbf{U}^T\underline{d}$$

show that we can write $\underline{x}$ as

$$\underline{x} \;=\; \mathbf{U}\alpha$$

where

$$\alpha_i \;=\; \frac{c_i^2\tilde{d}_i}{c_i^2 + bs_i^2}$$

b) Using the GSVD and the same method as demonstrated in class, show that we can write

$$\text{Trace}(I - A(b))^2 = N - \sum_{i=1}^{i=M} \frac{c_i^2}{c_i^2 + bs_i^2}$$

where

$$\begin{aligned} N &= \text{the number of columns in } \mathbf{U} \\ M &= \text{the number of } c_i \end{aligned}$$

2) For the three test problems from homework 5, with varying levels of noise, compare GCV, CRESO, maximum curvature, and GCV methods for choosing the value of b for **zero** order Tikhonov regularization. Plot the locations of b on an L-curve for each case. Specifically, you need to determine the chosen value of b, locate that b on the L-curve, and compare the estimated g values (what we are really trying to find) when the different b's are used. I do not expect you to write a real maximization routine, but try a range of b's (such as by using the **logspace** command) and find which of those b's gives the minimum value of $V(b)$. The chosen b's should not be at the edge of the L-curves.

<u>For your Cultural Benefit</u>

It can be shown that the corresponding GCV method for SVD is to choose the number of modes k in the SVD expansions to minimize the function

$$V(k) = \frac{||\underline{x}_{1:k} - \underline{d}||^2}{(N-k)^2}$$

Where N corresponds to the number of components in $\underline{x}$ and we have

$$\begin{aligned} \alpha &= \mathbf{U}^T \underline{d} \\ \underline{x}_{1:k} &= \mathbf{U} \alpha_{1:k} \end{aligned}$$

where $\alpha_{1:k}$ is equal to α for the first k components, while the remaining components are equal to zero.