Name Solutions Mail Box ________________

ECE-420
Exam 1
Fall 2015

Calculators and Laptops cannot be used. You must show your work to receive credit.

Problem 1	_________/10
Problem 2	_________/15
Problem 3	_________/12
Problem 4	_________/15
Problem 5	_________/8
Problem 6	_________/20
Problem 7	_________/20

Total _______________

Name____ Mail Box ________________

1a) Use long division to determine the first three nonzero terms in **the impulse response** for the following transfer function $H(z) = \dfrac{z+1}{z^2+z+1}$

$$\begin{array}{r|l} & z^{-1} - z^{-3} + z^{-4} \\ \hline z^2+z+1 & z+1 \\ & z+1+z^{-1} \\ \hline & -z^{-1} \\ & -z^{-1}-z^{-2}-z^{-3} \\ \hline & z^{-2}+z^{-3} \end{array}$$

$$\boxed{h(n) = \delta(n-1) - \delta(n-3) + \delta(n-4) + \cdots}$$

1b) Assume a system has impulse response $h(n) = (0.5)^n u(n)$ Determine the output if the input is $x(n) = 2\delta(n) - 3\delta(n-2)$

$$y(n) = h(n) * x(n) = h(n) * [2\delta(n) - 3\delta(n-2)]$$

$$= 2h(n) - 3h(n-2)$$

$$= \boxed{2(0.5)^n u(n) - 3(0.5)^{n-2} u(n-2) = y(n)}$$

Name____ Mail Box ________________

2) Assume we have a process which we expect to be able to model as $g(t) = \alpha t + \beta$. We perform an experiment and determine values for $g(t)$ for a number of different values of t, so we have the values

$$\begin{array}{ll} t_0, & g(t_0) \\ t_1, & g(t_1) \\ t_2, & g(t_2) \\ t_3, & g(t_3) \\ t_4, & g(t_4) \end{array}$$

We want to estimate α and β using a least squares type of approach, with the model $z = Aw$. Describe how you would do this. Be sure in your description, to indicate what the z, w, and A are in terms of the values given t_i, $g(t_i)$

$$g(t_i) = [t_i \quad 1]\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

$$z = \begin{bmatrix} g(t_0) \\ \vdots \\ g(t_4) \end{bmatrix} \qquad A = \begin{bmatrix} t_0 & 1 \\ \vdots & \vdots \\ t_4 & 1 \end{bmatrix} \qquad w = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

$$w = (A^T A)^{-1} A^T z$$

Name____ Mail Box ________________

3) Determine the rank of each of the following matrices (just write the number):

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ rank = 1

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$ rank = 2

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$ rank = 2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ rank = 1

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ rank = 2

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$ rank = 3

Name______________________________ Mail Box ______________

4) Consider the discrete-time state variable system,

$$x(n+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(n) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(n)$$

Assuming we start at the origin (the point 0,0) after 3 time steps we have

$$x(3) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = Au$$

a) If the input is $u^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ determine $x(3)$

$$x(3) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 3 \end{bmatrix} = x(3)}$$

b) Find a **unit vector** in the null space of the matrix A

$$\underline{n} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \qquad \hat{\underline{n}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

c) Is the input vector in step (a) the input with the smallest magnitude to go from the origin to the point $x(3)$? Explain your answer.

$\underline{u} \cdot \hat{\underline{n}} = 0$ no components of $\underline{u}$ in the null space

$\Rightarrow$ optimal

Name______________________________ Mail Box ______________

5) Use a Lagrange multiplier to find the point on the plane $2x+4y-6=0$ nearest the origin

$$H = x^2+y^2+\lambda(2x+4y-6)$$

$$\frac{\partial H}{\partial x} = 2x + 2\lambda = 0 \qquad x = -\lambda$$

$$\frac{\partial H}{\partial y} = 2y + 4\lambda = 0 \qquad y = -2\lambda$$

$$2x + 4y = 6 = 2(-\lambda) + 4(-2\lambda) = -10\lambda$$

$$\lambda = \frac{6}{-10} = \frac{-3}{5}$$

$$\boxed{x = 3/5 \quad y = \frac{6}{5}}$$

Name____ Mail Box ______________

6) Consider the overdetermined system $Ax=b$. Rather than just performing a normal least squares solution, we want to penalize the magnitude of the solution by some weighting scalar μ. Thus we want to find the value of x that minimizes

$$H = \| Ax-b \|^2 + \mu \| x \|^2 = (Ax-b)^T(Ax-b) + \mu (x^T I x)$$

a) Show that the solution to the above minimization problem is given by $x = (A^T A + \mu I)^{-1} A^T b$

b) Assume now we compute the singular value decomposition of A, $A = USV^T$. Recall that U and V are unitary matrices, so $U^T U = V^T V = I$. This means that $U^{-1} = U^T$ and $V^{-1} = V^T$. S is a diagonal matrix with elements σ_i on the diagonal (assume S has full rank and is invertible). Assume that we can write $b = U\beta$ and $x = V\alpha$. Determine how α and β are related. *Hint: it may be helpful to write* $I = VIV^T$

Hints: μ *is* ***not*** *a Lagrange multiplier, there are no Lagrange multipliers in this problem. You can do both parts of this problem independently.*

Cultural Point: This technique is known as zero order Tikhonov regularization.

a) $H = (x^T A^T - b^T)(Ax - b) + \mu x^T I x = x^T A^T A x - 2b^T A x + b^T b + \mu x^T I x$

$\frac{dH}{dx} = 2x^T A^T A - 2b^T A + 2\mu x^T I = 0$

$(A^T A + \mu I)x = A^T b$ $\quad \boxed{x = (A^T A + \mu I)^{-1} A^T b}$

b) $x = (A^T A + \mu I)^{-1} A^T b$

$(V\alpha) = [(USV^T)^T (USV^T) + \mu V V^T]^{-1} (USV^T)^T (U\beta)$

$V\alpha = [V S^T \underbrace{U^T U}_{1} S V^T + \mu V I V^T]^{-1} V S^T \underbrace{U^T U}_{1} \beta$

$= [V(S^T S + \mu I)V^T]^{-1} V S^T \beta$

$= V [S^T S + \mu I]^{-1} V^T V S^T \beta = V [S^T S + \mu I]^{-1} S^T \beta$

$\alpha = [S^T S + \mu I]^{-1} S^T \beta$

$$\boxed{\alpha_i = \frac{\sigma_i \beta_i}{\sigma_i^2 + \mu}}$$

Name____ Mail Box ____________________

7) For impulse response $h(n)=\left(\frac{1}{2}\right)^{n}u(n-2)$ and input $x(n)=\left(\frac{1}{2}\right)^{-n}u(-n+1)$, the system output can be written as $A(n)u(n-3)+B(n)u(2-n)$. Determine an expression for **both** $A(n)$ **and** $B(n)$. You do not need to simplify your expression but you must evaluate all sums. For my sanity (not that you care), evaluate the convolution using the form $y(n)=\sum_{k=-\infty}^{k=\infty}h(n-k)x(k)$

$$y(n)=\sum_{K=-\infty}^{\infty}\left(\tfrac{1}{2}\right)^{n-K}u(n-K-2)\left(\tfrac{1}{2}\right)^{-K}u(-K+1)$$

$$=\left(\tfrac{1}{2}\right)^{n}\sum_{K=-\infty}^{\infty}\left(\tfrac{1}{4}\right)^{-K}u(n-K-2)u(-K+1)$$

$u(n-K-2)=1$ for $n-K-2\geq 0$, $n-2\geq K$

$u(-K+1)=1$ for $-K+1\geq 0$, $1\geq K$

same for $n=3$

$n\geq 3$ $\quad A(n)=\left(\tfrac{1}{2}\right)^{n}\sum_{K=-\infty}^{1}\left(\tfrac{1}{4}\right)^{-K}$ $\quad$ let $l=1-K$ $\quad l-1=-K$

$$=\left(\tfrac{1}{2}\right)^{n}\sum_{l=0}^{\infty}\left(\tfrac{1}{4}\right)^{l}\left(\tfrac{1}{4}\right)^{-1}=4\left(\tfrac{1}{2}\right)^{n}\frac{1}{1-1/4}=4\left(\tfrac{1}{2}\right)^{n}\frac{1}{\frac{3}{4}}=\boxed{\frac{16}{3}\left(\tfrac{1}{2}\right)^{n}}$$

$n\leq 2$ $\quad B(n)=\left(\tfrac{1}{2}\right)^{n}\sum_{K=-\infty}^{n-2}\left(\tfrac{1}{4}\right)^{-K}$ $\quad$ let $l=n-2-K$, $\quad l-n+2=K$

$$=\left(\tfrac{1}{2}\right)^{n}\sum_{l=0}^{\infty}\left(\tfrac{1}{4}\right)^{l-n+2}=\left(\tfrac{1}{4}\right)^{2}\left(\tfrac{1}{2}\right)^{n}\left(\tfrac{1}{4}\right)^{-n}\frac{1}{1-1/4}$$

$$=\frac{1}{16}2^{n}\frac{4}{3}=\boxed{\frac{2^{n}}{12}}$$