ECE-420: Discrete-Time Control Systems
Homework 6

Due: Thursday October 24 in class
Exam 2, Friday October 25, 2013

1) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$$

where the explicit dependence of G and H on the sampling time T has been emphasized. Here

$$G(T) = e^{AT}$$
$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if A is invertible, we can write $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if A is invertible and T is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\dot{\underline{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = Ax(kT) + Bu(kT)$$

we get the same answer as in part **b,** but using this approximation we do not need to assume A is invertible.

d) Show that if we use two terms in the approximation for e^{AT} (and no assumptions about A being invertible), we can write the state equations as

$$\underline{x}(k+1) = \left[I + AT\right]\underline{x}(k) + [IT + \frac{1}{2}AT^2]Bu(k)$$

2) The Matlab script **homework6.m** computes the discrete-time equivalent state variable system from a continuous time system exactly, and then it uses a Taylor series approximation. However, the code is incomplete in that it does not use a Taylor series estimate for G.

a) Modify the code so that it uses a Taylor series estimate for G
b) Determine the minimum number of terms you think you need to produce a reasonable estimate of G (compared to Matlab's calculation)
c) Print and turn in the final plot
d) Use the second system (uncomment it), set the sampling interval to 0.01, the final time to 0.5, and repeat parts (a)-(c)

3) For the state variable system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

a) Show that

$$e^{AT} = \begin{bmatrix} 2e^{2T} - e^{3T} & e^{2T} - e^{3T} \\ 2e^{3T} - 2e^{2T} & 2e^{3T} - e^{2T} \end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system $\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$ for $T = 0.1$ (integrate each entry in the matrix $e^{A\lambda}$ separately.) Compare your answer with that given by Matlab's **c2d** command, [G,H] = c2d(A,B,Ts).

4) For the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

a) Find the eigenvalues and characteristic equation for A.

b) Determine an expression for A^2 in terms of A and I and then show explicitly that the matrix A satisfies its own characteristic equation by using the A matrix and evaluating both sides of the equation.

c) Using the Cayley-Hamilton method (matching on eigenvalues), show that $e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

d) Compute e^{At} using the Laplace transform method.

5) For the continuous time model

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-\tau)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t)$$

derive the equivalent ZOH discrete-time system for sampling interval T

$$\begin{bmatrix} \underline{x}([k+1]T) \\ u(kT) \end{bmatrix} = \begin{bmatrix} G & H_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}(kT) \\ u([k-1]T) \end{bmatrix} + \begin{bmatrix} H_0 \\ I \end{bmatrix} u(kT)$$

$$y(kT) = C \begin{bmatrix} \underline{x}(kT) \\ u([k-1]T) \end{bmatrix}$$

Specifically, you should show $G = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $H_0 = \begin{bmatrix} \frac{(T-\tau)^2}{2} \\ t-\tau \end{bmatrix}$, $H_1 = \begin{bmatrix} \tau(T - \frac{\tau}{2}) \\ \tau \end{bmatrix}$

6) For the continuous time model

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-0.03)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t)$$

derive the equivalent ZOH (zero order hold, this is our standard method of sampling) discrete-time system

$$\begin{bmatrix} \underline{x}([k+1]T) \\ u(kT) \end{bmatrix} = \begin{bmatrix} G & H_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}(kT) \\ u([k-1]T) \end{bmatrix} + \begin{bmatrix} H_0 \\ I \end{bmatrix} u(kT)$$

$$y(kT) = C \begin{bmatrix} \underline{x}(kT) \\ u([k-1]T) \end{bmatrix}$$

for $T = 0.1$. Specifically, determine G, H_0, H_1, and C. You should do all of the calculations in Matlab using the **c2d** command and the **expm** command. Assume we want the system output to remain the same.

7) Prove or disprove the following claims: if u, v, and w are linearly independent vectors, then so are

a) $u, u+v, u+v+w$
b) $u+2v-w, u-2v-w, 4v$
c) $u-v, v-w, w-u$
d) $-u+v+w, u-v+w, -u+v-w$

Note: You must do this for arbitrary vectors. ***Do Not*** assume $u, v,$ and w are specific vectors.

8) Determine the ranks of the following matrices. Do not use a calculator or computer, look for linearly independent columns or rows.

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$ d) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

e) $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ f) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Answers (scrambled): 3, 2, 2, 1, 2, 2

9) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0)=0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) After 1 time step we have $x(1) = Hu(0) = Mu(0)$ so $M = H$. After 2 time steps we have

$$x(2) = Gx(1) + Hu(1) = GHu(0) + Hu(1) = \begin{bmatrix} GH & H \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = M\tilde{u}(1)$$

so $M = \begin{bmatrix} GH & H \end{bmatrix}$ and $\tilde{u}(1) = \begin{bmatrix} u(0) & u(1) \end{bmatrix}^T$. Now assume we want $x(2) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Can you find an input vector $\tilde{u}(1)$, and hence input values $u(0)$ and $u(1)$, to make this happen? If you cannot go from the origin to any possible state in *at most* n steps, where $x(k) \in \mathbb{R}^n$, then the system is not controllable. Why at most n steps? See below…..

c) The **Cayley-Hamilton Theorem** from Linear Algebra states that *a matrix satisfies its own characteristic equation.* The characteristic equation of a matrix G is found by setting the determinant of $W = zI - G$ equal to zero. Show that the characteristic equation for our system is given by $z^2 - 1 = 0$ and then verify that $G^2 - I = 0$.

d) Now let's look at the third time step

$$x(3) = Gx(2) + Hu(2) = G^2Hu(0) + GHu(1) + Hu(2)$$

Using the Cayley-Hamilton Theorem, we can write $G^2 = I$. Show that we can then write

$$x(3) = \begin{bmatrix} GH & H \end{bmatrix} \tilde{u}(2)$$
$$\tilde{u}(2) = \begin{bmatrix} u(1) & u(2) + u(0) \end{bmatrix}^T$$

e) Show that we can write

$$x(4) = \begin{bmatrix} GH & H \end{bmatrix} \tilde{u}(3)$$
$$\tilde{u}(3) = \begin{bmatrix} u(0) + u(2) & u(1) + u(3) \end{bmatrix}^T$$

At this point, it should be clear that if we cannot find an input to go from the origin to a particular final state in $n = 2$ steps for a second order system we never will be able to get there, no matter how long we let the system run. If $x \in \mathbb{R}^n$, then the *controllability matrix* is defined to be $M = \begin{bmatrix} G^{n-1}H & G^{n-2}H & \ldots & GH & H \end{bmatrix}$. For a system to be controllable, this matrix must have rank n, or, equivalently, n linearly independent columns (or rows).

f) Now assume we are using state variable feedback with a *prefilter gain* G_{pf} , so $u(k) = G_{pf} r(k) - Kx(k)$. Here $r(k)$ is the *reference input* and $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is the *feedback gain matrix*. With this form of state variable feedback, we have the system

$$x(k+1) = Gx(k) + H\left[G_{pf} r(k) - Kx(k)\right] = [G - HK]x(k) + [HG_{pf}]r(k)$$

or

$$x(k+1) = \tilde{G}x(k) + \tilde{H}r(k)$$

Note that now the system input is the reference input $r(k)$. Show that for $D = 0$ the transfer matrix is given by

$$F(z) = \frac{Y(z)}{R(z)} = C(zI - \tilde{G})^{-1}\tilde{H} = \frac{G_{pf}(z+1)}{(z+k_1)(z+k_2) - (k_1 - 1)(k_2 - 1)}$$

g) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

h) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

In summary, if the system is controllable

- *We can go from the origin to any final state in n steps (the rank of the controllability matrix M is n)*

- *We can place the poles of the closed loop system anywhere we want using state variable feedback*

10) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Find the M matrix after two time steps. Now assume we want $x(2) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Can you find an input vector $\tilde{u}(1)$, and hence input values $u(0)$ and $u(1)$, to make this happen?

c) Show that the characteristic equation for G is given by $z^2 - 2z + 1 = 0$ and verify that $G^2 = 2G - I$.

d) Show that we can then write

$$x(3) = \begin{bmatrix} GH & H \end{bmatrix} \tilde{u}(2)$$
$$\tilde{u}(2) = \begin{bmatrix} 2u(0) + u(1) & u(2) - u(0) \end{bmatrix}^T$$

e) Show that we can write

$$x(4) = \begin{bmatrix} GH & H \end{bmatrix} \tilde{u}(3)$$
$$\tilde{u}(3) = \begin{bmatrix} 3u(0) + 2u(1) + u(2) & -2u(0) - u(1) + u(3) \end{bmatrix}^T$$

f) Now assume we are using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$. Show that for $D = 0$ the transfer matrix is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}(z-1)}{(z-1)(z+k_2-1)}$$

g) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

h) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

11) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Find the M matrix after two time steps. Now assume we want $x(2) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Can you find an input vector $\tilde{u}(1)$, and hence input values $u(0)$ and $u(1)$, to make this happen? Now assume we want $x(2) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Can you find an input vector $\tilde{u}(1)$ to make this happen?

c) Show that the characteristic equation for G is given by $z^2 - z - 1 = 0$ and verify that $G^2 = G + I$.

d) Show that we can then write

$$x(3) = \begin{bmatrix} GH & H \end{bmatrix} \tilde{u}(2)$$
$$\tilde{u}(2) = \begin{bmatrix} u(0)+u(1) & u(0)+u(2) \end{bmatrix}^T$$

e) Show that we can write

$$x(4) = \begin{bmatrix} GH & H \end{bmatrix} \tilde{u}(3)$$
$$\tilde{u}(3) = \begin{bmatrix} 2u(0)+u(1)+u(2) & u(0)+u(1)+u(3) \end{bmatrix}^T$$

f) Now assume we are using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$. Show that for $D = 0$ the transfer matrix is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}}{z^2 + (k_2 - 1)z + (k_1 - 1)}$$

g) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

h) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero? If we want the poles to be at p_1 and p_2 show that $k_2 = 1 - (p_1 + p_2)$ and $k_1 = 1 + p_1 p_2$.