ECE-420: Discrete-Time Control Systems
Homework 4

Due: Friday October 4 at the beginning of class

1) For this problem, assume $\underline{p} = \begin{bmatrix} a \\ b \end{bmatrix}, \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and show the following:

a) for $f(\underline{x}) = \underline{p}^T \underline{x}, \dfrac{df}{d\underline{x}} = \underline{p}^T$

b) for $f(\underline{x}) = \underline{x}^T \underline{p}, \dfrac{df}{d\underline{x}} = \underline{p}^T$

c) for $f(\underline{x}) = A\underline{x}, \dfrac{df}{d\underline{x}} = A$

d) for $f(\underline{x}) = A^T \underline{x}, \dfrac{df}{d\underline{x}} = A^T$

e) for $f(\underline{x}) = \underline{x}^T A\underline{x}, \dfrac{df}{d\underline{x}} = \underline{x}^T (A + A^T)$

2) The error vector $\underline{e}$ between observation vector $\underline{d}$ and the estimate of the input $\hat{\underline{x}}$ is $\underline{e} = \underline{d} - A\hat{\underline{x}}$. We want to weight the errors by a *symmetric* matrix R. Find $\hat{\underline{x}}$ to minimize $\underline{e}^T R \underline{e}$. (This is a weighted least squares.)

3) Assume we expect a process to follow the equation $y(t) = \dfrac{1}{ct + d\sqrt{t}}$, and we have measurements:

t	1.0	2.0	3.0	4.0	5.0	6.0
$y(t)$	0.30	0.21	0.14	0.12	0.11	0.09

a) Determine a least squares estimate of the parameters c and d.

b) Using your least squares estimate of the parameters, estimate the value of $y(2.5)$.

c) Assume we want to use a weighted least squares with r = diag([2 2 2 1 1 1]). Determine the weighted least squares estimate of parameters c and d.

4) Assume we expect a process to follow the following equation: $\gamma(x) = \varepsilon e^{\beta x}$

Assume we measure $\gamma(x)$ at various locations x:

x	0.0	0.1	0.4	2.0	4.0
$\gamma(x)$	2.45	2.38	2.30	1.40	0.70

a) Determine a least squares fit to the parameters ε and β. (*Hint: Try logarithms.*)

b) Estimate $\gamma(3.0)$

5) Assume we have an experimental process we are modeling, and, based on sound physical principles, we believe the relationship between x and y to be $y(x) = \left(\dfrac{\alpha}{x}\right)^{\beta}$ and we have the following measurements:

x	1.0	2.0	3.0	4.0
$y(x)$	8.0	1.0	0.1	0.1

a) Find a least squares estimate for α and β. (Hint: You cannot solve for α directly. Let $w = \beta \log(\alpha)$, solve for w and β, and then infer α.)

b) Estimate $y(3.5)$.

6) For this problem you will need the file homework4.m. This file just introduces you to the value of Matlab's pinv command. The file demonstrates trying to find the input signal given the output signal and a transfer matrix. Initially there is a very large signal to noise ratio (SNR) of 1000 dB. You are to modify the **pinv** command to get a reasonable estimate when the SNR is 10 dB and 20 dB. Do not expect to be able to get the exact answer, just try and get reasonably close. Turn in your estimates for x for the 10 and 20 dB SNR values, and indicate how you modified the **pinv** command. Since we are dealing with noise, you should run the program five times for each case and report all of the results. (you will get different results each time you run the program once the signal to noise level gets smaller.)

7) Consider the following discrete-time state variable system

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) = Gx(k) + Hu(k)$$

with the initial condition $x(0) = 0$

a) Show that after three time steps we get the system of equations

$$x(3) = G^2 Hu(0) + GHu(1) + Hu(2) = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

b) Show that if the input is $u(0) = 1, \ u(1) = 0, u(2) = 1$ we end up at the final state $x(3) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

c) Determine an optimal (minimum norm) control signal that will solve the underdetermined system and take the system from $x(0) = 0$ to $x(3) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

8) (Oh No, a real world problem….) The primary cause of cardiac arrhythmias is unusual electrical conduction patterns in the heart, often caused by congenital defects or heart disease. One way cardiologists try to determine areas of unusual conduction is to map electrical patterns in the heart (or on the various surfaces of the heart chambers). One such method involves inserting two or more catheters through a vein in the leg into the left ventricle (LV). A special tip on one of the catheters is inflated (like a balloon), and there are 64 electrodes on this balloon. Based on measurements on these electrodes, the electrical pattern on the LV endocardium (inner heart surface) is estimated. See Figure 1.
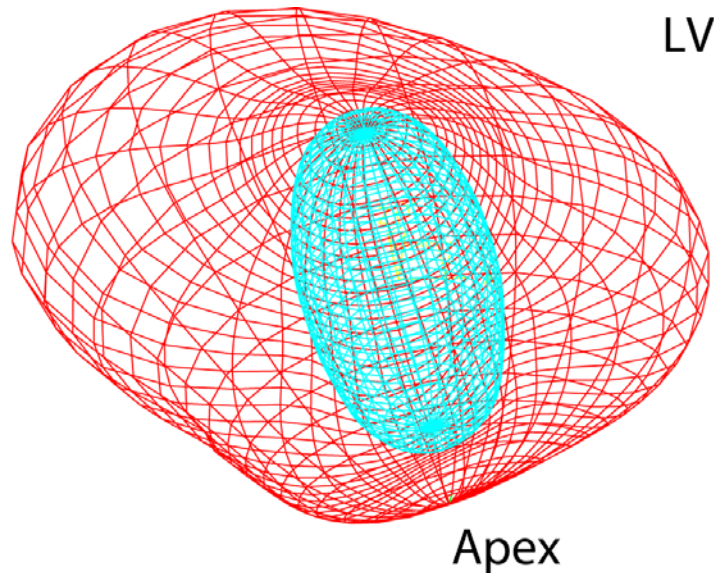
Figure 1. Schematic of the left ventricle and an inflated balloon sensor.

Unfortunately, even though the distance from the source of the electricity (the endocardium) to the electrodes is at most a few centimeters, the transfer matrix relating them has a condition number of approximately one million. In this problem we know the measured electrical potentials on the probe (y), and the transfer matrix (H), and are trying to determine the actual potentials on the endocardium (x). One method to verify that we are doing this somewhat correctly is to use a separate catheter to actually measure the endocardial potentials at a few selected points and compare these points to our estimates. The program **probe_tsvd.m** use truncated singular value decomposition to estimate the potentials on the endocardium at selected points, and compare them with actual measurements. The input to this program is the number of terms to use in the svd, and the output is a plot of the estimated (blue) and measured (red) electrograms at various points on the endocardium and the correlation coefficients between the measured and estimated electrograms. The program also outputs a plot of the singular values. You need to try and determine the number of terms to use in the svd to get reasonable estimates, and turn in the resulting plot.

9) For our regular least squares problem, we are trying to find the value of $\hat{x}$ to minimize $H = \|d - A\hat{x}\|^2$ Sometimes this does not work very well and a more appropriate method is something call Tikhonov regularization, which is commonly used in solving *inverse problems*, which is really what we are doing. Loosely speaking, and inverse problem is when you know the output of a system and want to know the input when the system is not really *invertible*. For Tikhonov regularization, consider minimizing the following:

$$H = \|d - A\hat{x}\|^2 + \mu\|R\hat{x}\|^2$$

There are now two terms. The first term is from the regular least squares problem, and is what we would use to fit the system to the measured data. The second term imposes a penalty of some sort on the estimate. Here R is called the regularization operator. For *zero order* Tikhonov regularization R is an identity matrix and the amplitude of the estimate $\hat{x}$ is penalized. For *first order* Tikhonov regularization R is a gradient operator and the first derivative of the estimate $\hat{x}$ is penalized. For *second order* Tikhonov regularization R is a Laplacian operator and the second derivative of the estimate $\hat{x}$ is penalized. Clearly, the higher the order of the regularization operator, the smoother the estimate will be. Finally, μ is the regularization parameter, and it indicates the relative weight between the two terms (matching the measured data and a smooth estimate).

a) Carry out the above minimization and show that the solution is $\hat{x} = \left(A^T A + \mu R^T R\right)^{-1} A^T d$.

b) Just as we can use Singular Value Decomposition (SVD) for our least squares problem, we can use Generalized Singular Value Decomposition (GSVD) for this problem. For GSVD we can write

$$A = UCX^T$$
$$R = VSX^T$$

Where U and V are *unitary* matrices ($U^T U = V^T V = I$), C and S are *diagonal* matrices with elements c_i and s_i on the diagonal, and $C^T C + S^T S = I$. Assuming X^{-1} and X^{-T} exist, and defining

$$\tilde{d} = U^T d$$
$$\tilde{x} = X^T \hat{x}$$
$$\hat{x} = X^{-T} \tilde{x}$$

Show that we can write

$$\tilde{x}_i = \frac{c_i \tilde{d}_i}{c_i^2 + \mu s_i^2}$$

Although Matlab has routines to compute the GSVD, it is actually a bit more involved than it appears here to actually implement this.

c) The Matlab code **probe_tikk.m** implements first order Tikhonov regularlization for the probe data used in problem 7. There is one argument to this program, and that is μ, the regularization parameter. If $\mu = 0$ we will get the usual least squares (garbage) solution. You need to try and determine a good value of μ to use to get reasonable estimates, and turn in the resulting plot.

Note: it should be pointed out that in a real inverse problem, the actual solution is not known and the values of the various parameters must be chosen in some other way. We are seriously cheating!