ECE-420: Discrete-Time Control Systems
Homework 2

Due: Friday September 14 at the beginning of class

1) For the z -transform

$$X(z) = \frac{3}{z-2}$$

a) Show that, by multiplying and dividing by z and then using partial fractions, the corresponding discrete-time sequence is

$$x(k) = -\frac{3}{2}\delta(k) + \frac{3}{2}2^k u(k)$$

b) By starting with the z -transform

$$G(z) = \frac{3z}{z-2}$$

where $Y(z) = z^{-1}G(z)$ and the z -transform properties, show that

$$x(k) = 3\,2^{k-1}u(k-1)$$

2) For the following transfer functions, use **long division** to determine estimates of the first few terms of the impulse responses

a) $H(z) = \dfrac{z^3 + 2z^2 + 3z + 2}{z+2}$

b) $H(z) = \dfrac{z^2 + 2z + 1}{z^3 + 3z}$

Answers: $h(n) = \delta(n+2) + 3\delta(n) - 4\delta(n-1) + 8\delta(n-2) - 16\delta(n-3) + ...$

$h(n) = \delta(n-1) + 2\delta(n-2) - 2\delta(n-3) - 6\delta(n-4) + ...$

3) For impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n-2)$ and input $x(n) = \left(\frac{1}{3}\right)^n u(n-2)$, use z-transforms of the input and impulse response to show the system output is $y(n) = \frac{1}{6}\left[\left(\frac{1}{2}\right)^{n-3} - \left(\frac{1}{3}\right)^{n-3}\right]u(n-3)$

Hint: Assume $Y(z) = z^{-3}G(z)$, determine $g(n)$ and then $y(n)$

4) For impulse response $h(n)=\left(\frac{1}{3}\right)^{n+1}u(n-1)$ and input $x(n)=\left(\frac{1}{2}\right)^{n-1}u(n-2)$, use z-transforms of the input and impulse response to show the system output is $y(n)=\frac{1}{3}\left[\left(\frac{1}{2}\right)^{n-2}-\left(\frac{1}{3}\right)^{n-2}\right]u(n-2)$

Hint: Assume $Y(z)=z^{-2}G(z)$, determine $g(n)$ and then $y(n)$

5) For impulse response $h(n)=\left(\frac{1}{2}\right)^{n-1}u(n)$ and input $x(n)=\left(\frac{1}{5}\right)^{n-2}u(n-3)$, use z-transforms of the input and impulse response to show the system output is $y(n)=\frac{4}{3}\left[\left(\frac{1}{2}\right)^{n-2}-\left(\frac{1}{5}\right)^{n-2}\right]u(n-2)$

Hint: Assume $Y(z)=z^{-2}G(z)$, determine $g(n)$ and then $y(n)$

6) In class we derived the relation ships

$$G(z)=\frac{rz[z\cos(\theta)-\gamma\cos(\beta-\theta)]}{z^2-2\gamma\cos(\beta)z+\gamma^2}=\frac{Az^2+Bz}{z^2+2Cz+\gamma^2}\leftrightarrow g(n)=r\gamma^n\cos(\beta n+\theta)u(n)$$

with

$$\beta=\cos^{-1}\left(\frac{-C}{\gamma}\right),\ \theta=\tan^{-1}\left(\frac{CA-B}{A\sqrt{\gamma^2-C^2}}\right),\ r=\sqrt{\frac{A^2\gamma^2+B^2-2ABC}{\gamma^2-C^2}}$$

a) Us the above formulas to find the impulse response $g(n)$ for $G(z)=\frac{z^2+0.5z}{z^2+0.2z+0.125}$.

b) Compute $g(n)$ from part **a** for $n=0,1,2,3,4$ and then perform long division to verify that your answer to **a** is correct for these terms

c) Determine the unit step response $y(n)$ for $G(z)=\frac{1}{z^2+0.1z+4}$, by using the form

$$\frac{Y(z)}{z}=\frac{1}{(z-1)(z^2+0.1z+4)}=\frac{\alpha_1}{z-1}+\frac{\alpha_2 z+\alpha_3}{z^2+0.1z+4}$$

Hint: α_1 can be found using the cover-up method, α_2 can be found by multiplying both sides by z and letting $z\rightarrow\infty$, and α_3 can be found by substituting a convenient value for z, like $z=0$.

d) Compute $y(n)$ from part **e** for $n=0,1,2,3,4$ and then perform long division to verify that your answer to **e** is correct for these terms

7) Consider the following difference equation

$$x(k+2)-4x(k+1)+4x(k)= f(k)$$

Assume all initial conditions are zero.

a) Determine the *impulse response* of the system, i.e., the response $x(k)$ when $f(k)=\delta(k)$.

b) Determine $h(0), h(1), h(2), h(3),$ and $h(4)$ from your answer to **a.** Compare this answer with the known values of $h(0)$ and $h(1)$. Using the difference equation compute $h(3)$ and $h(4)$ and compare these values to those in your solution (they should be the same!)

c) Determine the *step response* of the system, i.e., the response $x(k)$ when $f(k)=u(k)$

d) Determine $x(0), x(1), x(2), x(3),$ and $x(4)$ from your answer to **c.** Compare this answer with the known values of $x(0)$ and $x(1)$. Using the difference equation compute $x(3)$ and $x(4)$ and compare these values to those in your solution.

8) Consider the difference equation

$$x(k+2)-5x(k+1)+6x(k)= f(k)$$

where $f(k)=u(k)$, a unit step. Assume $x(0)=1$ and $x(1)=1$.

a) Determine the *Zero Input Response* (ZIR), $x_{ZIR}(k)$. This is the part of the solution $x(k)$ due to the initial conditions alone (assume the input is zero).

b) Determine the *Zero State Response* (ZSR), $x_{ZSR}(k)$. This is the part of the solution $x(k)$ due to the input alone (assume all initial conditions are zero).

c) Find the total response $x(k)=x_{ZIR}(k)+x_{ZSR}(k)$

d) Find the transfer function and the impulse response.

e) Determine $x(0), x(1), x(2), x(3),$ and $x(4)$ from your answer to **c.** Compare this answer with the known values of $x(0)$ and $x(1)$. Using the difference equation compute $x(3)$ and $x(4)$ and compare these values to those in your solution.