ECE-320: Linear Control Systems
Homework 9

Due: Tuesday February 16 at the beginning of class

1) Consider the discrete-time state variable model $\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$

where the explicit dependence of G and H on the sampling time T has been emphasized. Here

$$G(T) = e^{AT}$$

$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if A is invertible, we can write $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if A is invertible and T is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\dot{\underline{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = Ax(kT) + Bu(kT)$$

we get the same answer as in part **b,** but using this approximation we do not need to assume A is invertible.

d) Show that if we use two terms in the approximation for e^{AT} (and no assumptions about A being invertible), we can write the state equations as

$$\underline{x}(k+1) = \left[I + AT\right]\underline{x}(k) + [IT + \frac{1}{2}AT^2]Bu(k)$$

2) The Matlab script **homework9.m** computes the discrete-time equivalent state variable system from a continuous time system exactly, and then it uses a Taylor series approximation. However, the code is incomplete in that it does not use a Taylor series estimate for G.

a) Modify the code so that it uses a Taylor series estimate for G
b) Determine the minimum number of terms you think you need to produce a reasonable estimate of G (compared to Matlab's calculation)
c) Print and turn in the final plot
d) Use the second system (uncomment it), set the sampling interval to 0.01, the final time to 0.5, and repeat parts (a)-(c)

3) For the state variable system

$$\underline{\dot{x}}(t)=\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\underline{x}(t)+\begin{bmatrix}1\\ 1\end{bmatrix}u(t)$$

a) Show that

$$e^{AT}=\begin{bmatrix}2e^{2T}-e^{3T} & e^{2T}-e^{3T}\\ 2e^{3T}-2e^{2T} & 2e^{3T}-e^{2T}\end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system

$$\underline{x}(k+1)=G\underline{x}(k)+Hu(k)$$

for $T=0.1$ (integrate each entry in the matrix $e^{A\lambda}$ separately.) Compare your answer with that given by Matlab's **c2d** command, [G,H] = c2d(A,B,T).

4) For the matrix $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ show that $e^{At}=\begin{bmatrix}e^{t} & 0\\ te^{t} & e^{t}\end{bmatrix}$

5) Consider the discrete-time state variable model $\underline{x}(k+1)=G\underline{x}(k)+Hu(k)$ with the initial state $x(0)=0$. Let

$$G=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, H=\begin{bmatrix}1\\ 1\end{bmatrix}, C=\begin{bmatrix}1 & 0\end{bmatrix}, D=0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k)=G_{pf}r(k)-Kx(k)$ show that the transfer function is given by

$$F(z)=\frac{Y(z)}{R(z)}=C(zI-\tilde{G})^{-1}\tilde{H}=\frac{G_{pf}(z+1)}{(z+k_1)(z+k_2)-(k_1-1)(k_2-1)}$$

c) Show that if $G_{pf}=1$ and $k_1=k_2=0$, the transfer function reduces to that found in part **a.**

d) Is the system controllable? That is, is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

6) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$ show that the transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}(z-1)}{(z-1)(z+k_2-1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is the system controllable?