ECE-320: Linear Control Systems
Homework 6

Due: Tuesday January 26 at the beginning of class

1) Consider a system with closed loop transfer function $G_o(s) = \frac{\alpha k_p}{s+\alpha+k_p}$. The *nominal* values for the parameters are $k_p = 1$ and $\alpha = 2$.

a) Determine an expression for the sensitivity of the closed loop system to variations in k_p. Your final answer should be written as numbers and the complex variable *s*.
b) Determine an expression for the sensitivity of the closed loop system to variations in α. Your final answer should be written as numbers and the complex variable *s*.
c) Determine expressions for the *magnitude* of the sensitivity functions in terms of frequency, ω
d) As $\omega \to \infty$ the system is more sensitive to which of the two parameters?

2) Consider the plant

$$G_p(s) = \frac{\alpha_0}{s+\alpha_1} = \frac{3}{s+0.5}$$

where 3 is the nominal value of α_0 and 0.5 is the nominal value of α_1. In this problem we will investigate the sensitivity of closed loop systems with various types of controllers to these two parameters. We will assume we want the settling time of our system to be 0.5 seconds and the steady state error for a unit step input to be less than 0.1.

a) (*ITAE Model Matching*) Since this is a first order system, we will use the first order ITAE model,

$$G_o(s) = \frac{\omega_o}{s+\omega_o}$$

i) For what value of ω_o will we meet the settling time requirements and the steady state error requirements?

ii) Determine the corresponding controller $G_c(s)$.

iii) Show that the closed loop transfer function (using the parameterized form of $G_p(s)$ and the controller from part ii) is

$$G_o(s) = \frac{\frac{8}{3}\alpha_0(s+0.5)}{s(s+\alpha_1)+\frac{8}{3}\alpha_0(s+0.5)}$$

iv) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by $S_{\alpha_0}^{G_0} = \frac{s}{s+8}$

v) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by $S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2 + 8.5s + 4}$

b) (*Proportional Control*) Consider a proportional controller, with $k_p = 2.5$.

i) Show that the closed loop transfer function is $G_o(s) = \frac{2.5\alpha_0}{s + \alpha_1 + 2.5\alpha_0}$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by $S_{\alpha_0}^{G_0} = \frac{s + 0.5}{s + 8}$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by $S_{\alpha_1}^{G_o} = \frac{-0.5}{s + 8}$

c) (*Proportional+Integral Control*) Consider a PI controller with $k_p = 4$ and $k_i = 40$.

i) Show that the closed loop transfer function is $G_o(s) = \frac{4\alpha_0(s+10)}{s(s+\alpha_1) + 4\alpha_0(s+10)}$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by $S_{\alpha_0}^{G_0} = \frac{s(s+0.5)}{s^2 + 12.5s + 120}$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by $S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2 + 12.5s + 120}$

d) Using Matlab, simulate the unit step response of each type of controller. Plot all responses on one graph. Use different line types and a legend. Turn in your plot and code. ***Do not*** *make separate graphs for each system!*

e) Using Matlab and subplot, plot the sensitivity to α_0 for each type of controller on ***one graph*** at the top of the page, and the sensitivity to α_1 on one graph on the bottom of the page. Be sure to use different line types and a legend. Turn in your plot and code. Only plot up to about 8 Hz (50 rad/sec) using a semilog scale with the sensitivity in dB (see next page). ***Do not*** *make separate graphs for each system!*

In particular, these results should show you that the model matching method, which essentially tries and cancel the plant, are generally more sensitive to getting the plant parameters correct than the PI controller for low frequencies. However, for higher frequencies the methods are all about the same.

Hint: If $T(s) = \frac{2s}{s^2 + 2s + 10}$, *plot the magnitude of the frequency response using:*

```
T = tf([2 0],[1 2 10]);
w = logspace(-1,1.7,1000);
[M,P]= bode(T,w);
Mdb = 20*log10(M(:));
semilogx(w,Mdb); grid;
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xlabel('Frequency (rad/sec)');
ylabel('dB');

3) Assume $x(t) = 3 + 2\cos(2t - 3)$ is the input to an LTI system with transfer function

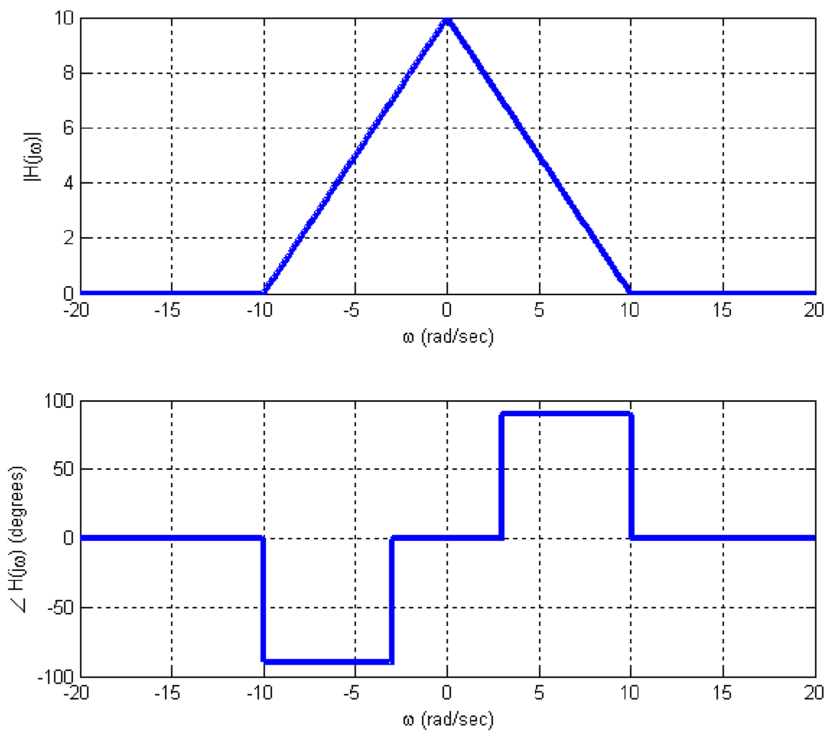
$$H(j\omega) = \begin{cases} 2e^{-j\omega} & |\omega| < 3 \\ 3e^{-j2\omega} & |\omega| \ge 3 \end{cases}$$

The **steady state output** will be

a) $y(t) = 6 + 4\cos(2t - 5)$ b) $y(t) = 4\cos(2t - 5)$ c) $y(t) = [3 + 2\cos(2t - 3)][2e^{-j\omega}]$

d) $y(t) = 6 + 4\cos(2t - 3)e^{-j2}$ e) $y(t) = 3 + 4\cos(2t - 5)$ f) none of these

4) Assume $x(t) = 2 + \sin(5t) + 3\cos(8t + 30^o)$ is the input to an LTI system with transfer function shown below



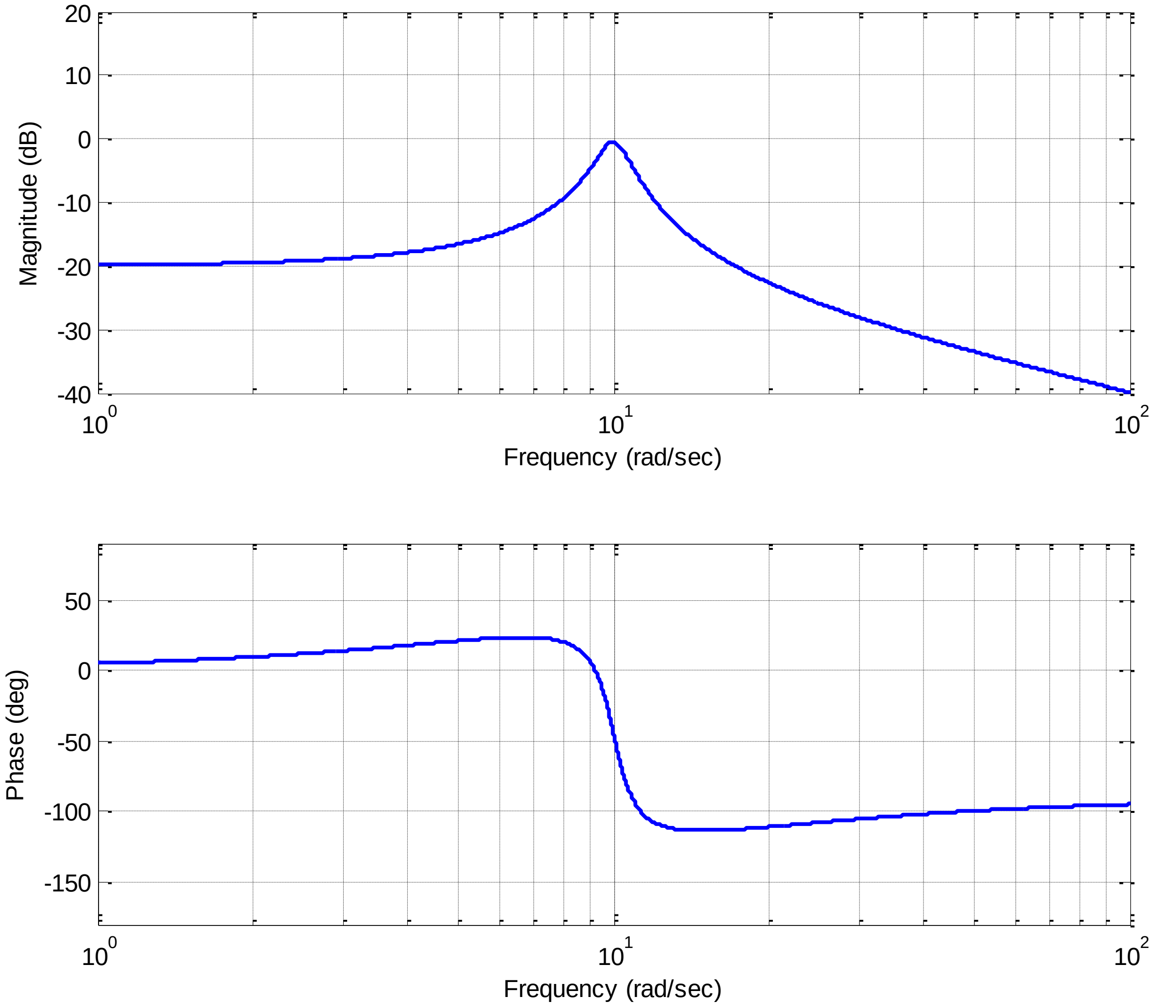
The **steady state output** of this system will be

a) $y(t) = 20 + 5\sin(5t + 90^o) + 6\cos(8t + 90^o)$ b) $y(t) = 2 + 5\sin(5t + 90^o) + 6\cos(8t + 90^o)$

c) $y(t) = 20 + 5\sin(5t + 90^o) + 6\cos(8t + 120^o)$ d) $y(t) = 10 + 5\sin(5t + 90^o) + 6\cos(8t + 120^o)$

e) none of these

Problems 5 and 6 refer to a system whose frequency response is represented by the Bode plot below



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5) If the input to the system is $x(t) = 5\cos(10t + 30^o)$, then the steady state output is best estimated as

a) $y_{ss}(t) = 0$ b) $y_{ss}(t) = 5\cos(10t + 30^o)$

c) $y_{ss}(t) = 5\cos(10t - 20^o)$ d) $y_{ss}(t) = 5\cos(10t - 50^o)$

6) If the input to the system is $x(t) = 50\sin(100t)$, then the steady state output is best estimated as

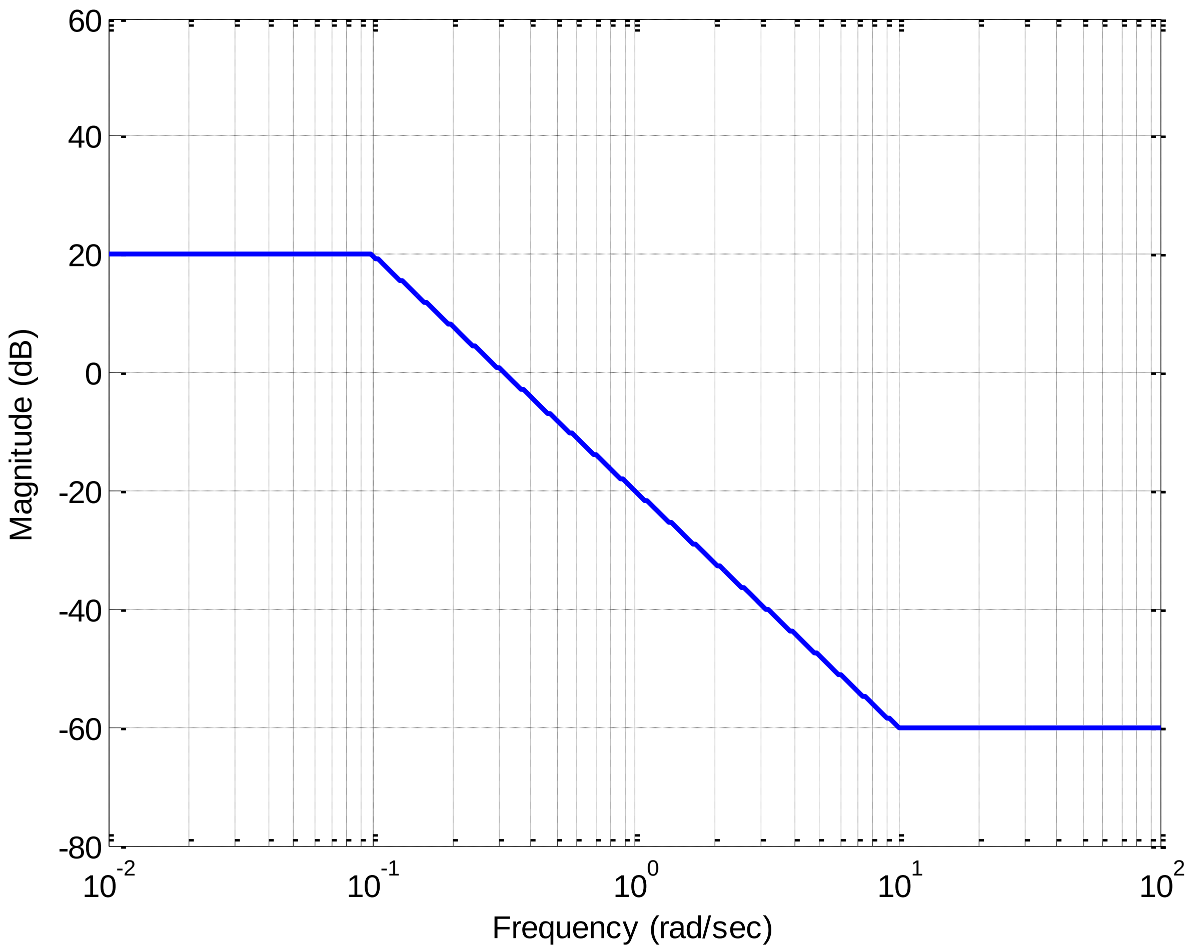
a) $y_{ss}(t) = -2000\sin(100t - 100^o)$ b) $y_{ss}(t) = 0.5\sin(100t - 100^o)$

c) $y_{ss}(t) = 2000\sin(100t - 100^o)$ d) $y_{ss}(t) = 5\sin(100t - 100^o)$

7) For the straight line approximation to the magnitude portion of a Bode plot shown below, the best estimate of the corresponding transfer function is

a) $H(s) = \dfrac{20\left(\frac{1}{10}s+1\right)}{10s+1}$ b) $H(s) = \dfrac{10\left(\frac{1}{10}s+1\right)}{10s+1}$

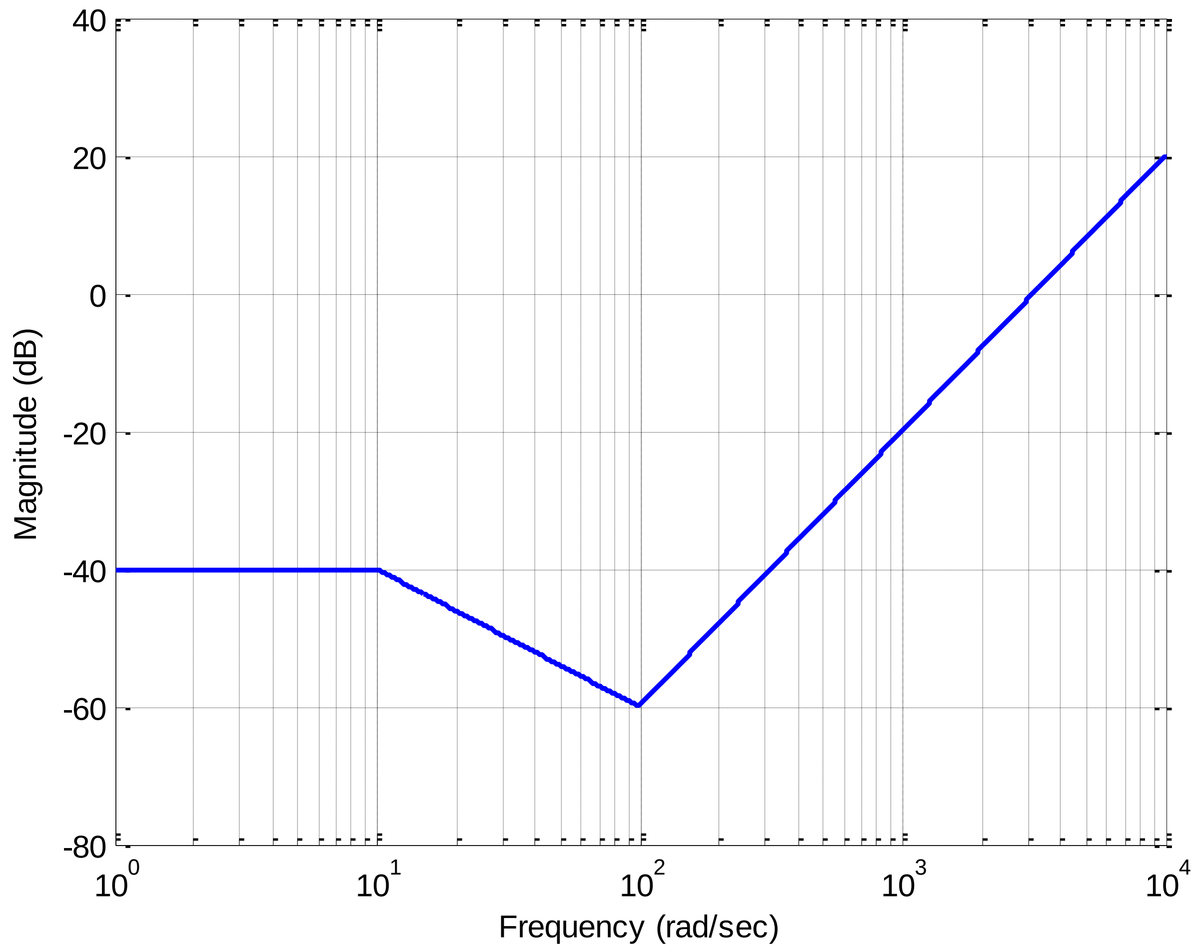
c) $H(s) = \dfrac{10\left(\frac{1}{10}s+1\right)}{(10s+1)^2}$ d) $H(s) = \dfrac{10\left(\frac{1}{10}s+1\right)^2}{(10s+1)^2}$



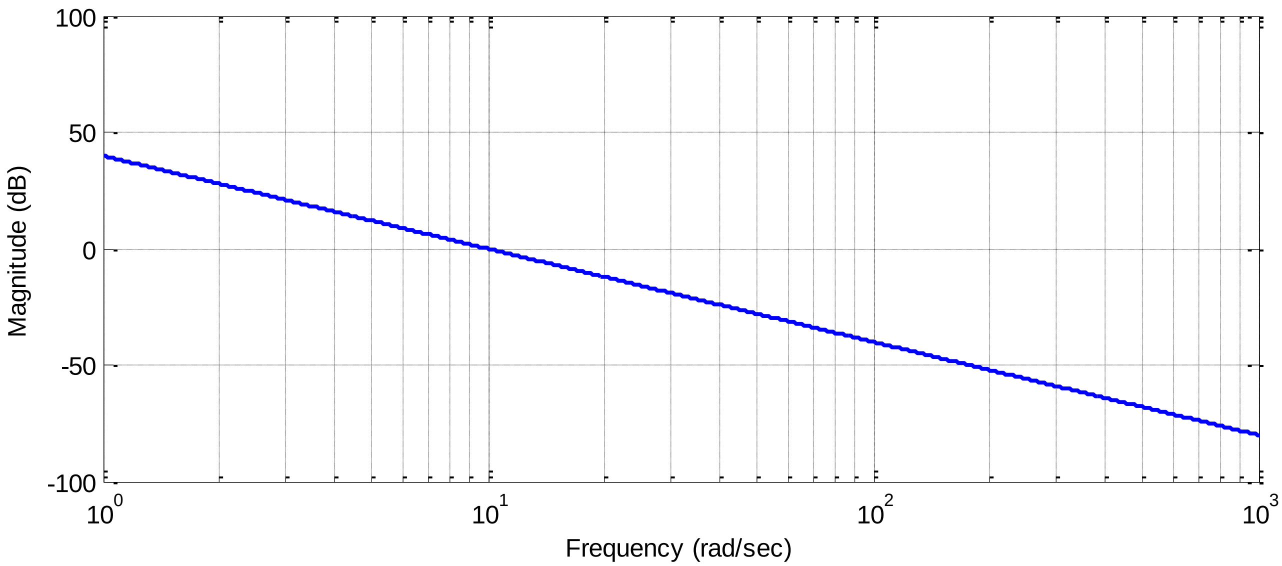
8) For the straight line approximation to the magnitude portion of a Bode plot shown below, the best estimate of the corresponding transfer function is

a) $H(s) = \dfrac{0.01\left(\frac{1}{100}s+1\right)^2}{\left(\frac{1}{10}s+1\right)}$ b) $H(s) = \dfrac{-40\left(\frac{1}{100}s+1\right)^2}{\left(\frac{1}{10}s+1\right)}$

c) $H(s) = \dfrac{0.01\left(\frac{1}{100}s+1\right)^3}{\left(\frac{1}{10}s+1\right)}$ d) $H(s) = \dfrac{0.01\left(\frac{1}{100}s+1\right)^3}{\left(\frac{1}{10}s+1\right)^2}$



9) The following three figures display the magnitude of six transfer functions. All of the poles and zeros of these transfer functions are in the left half plane (these are minimum phase transfer functions). All of the magnitudes, poles, and zeros are either zero or simple powers of 10 ($10^{-1}, 1, 10^{1}, 10^{2}$ etc). Estimate the transfer functions.



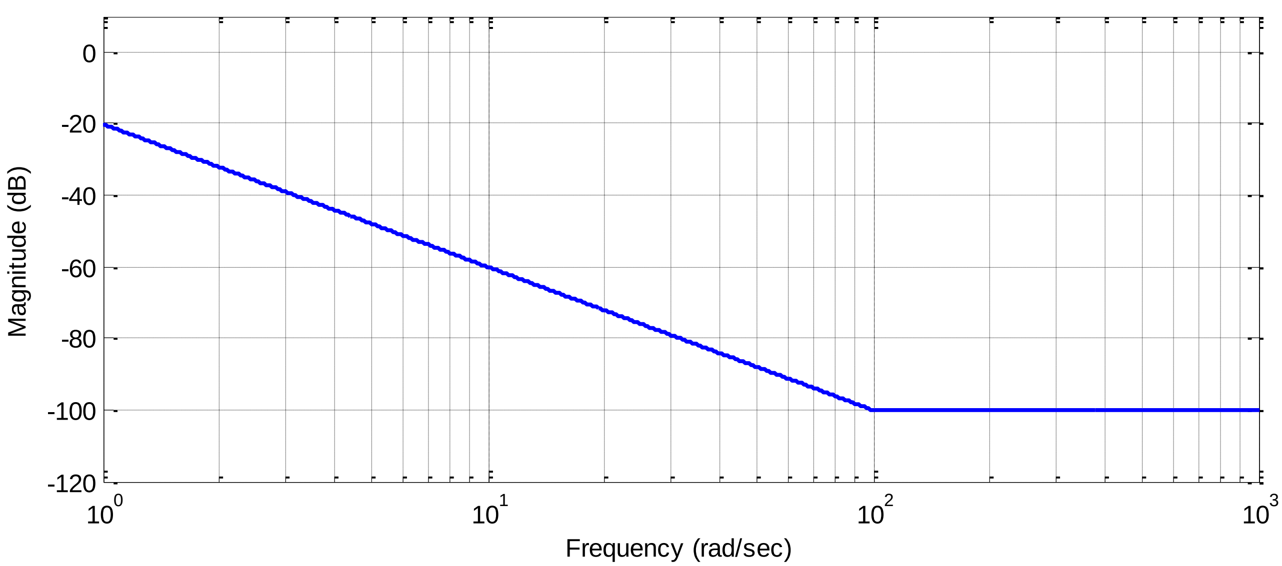


Figure 1: Problem 9, Systems a and b

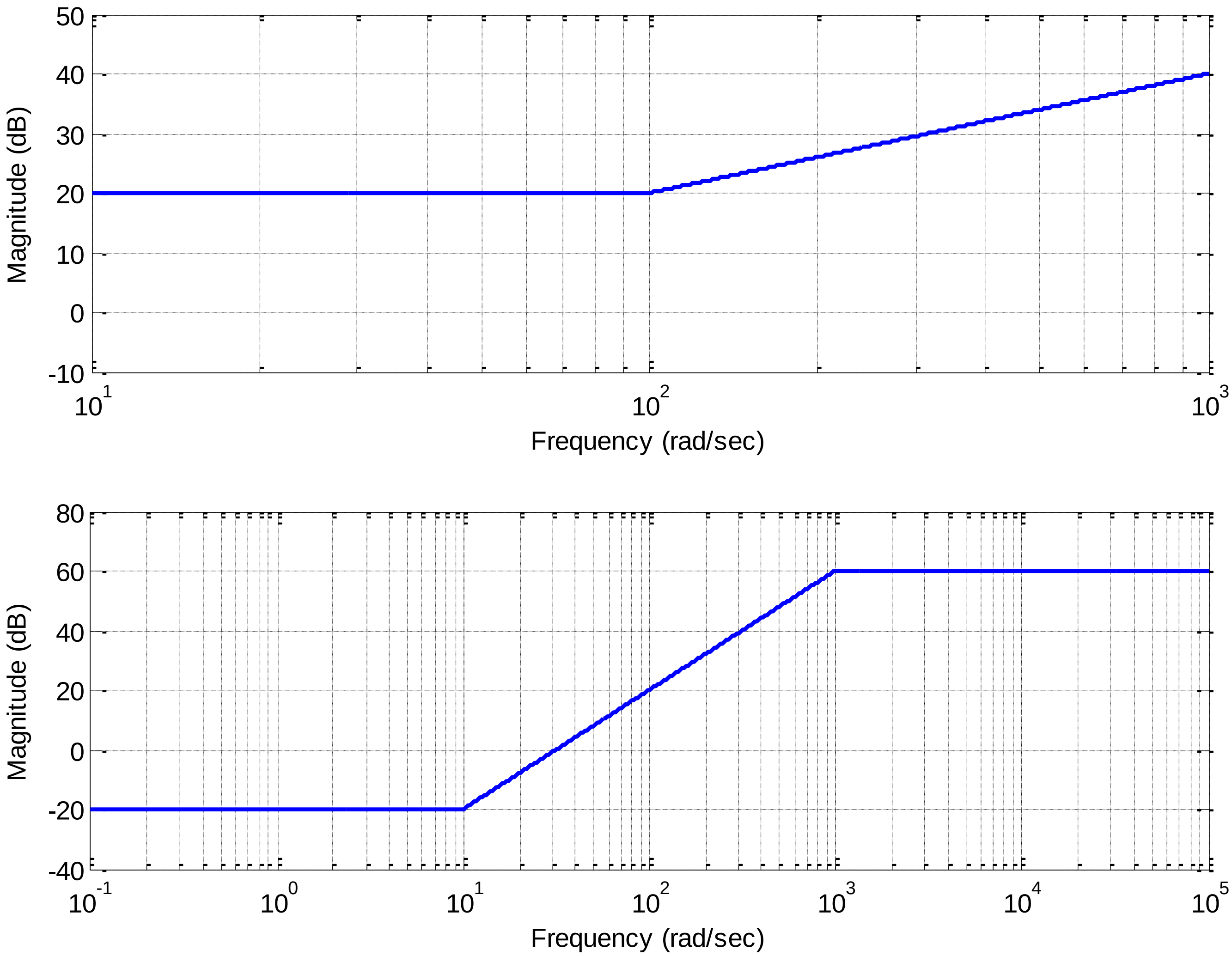


Figure 2: Problem 9, Systems c and d

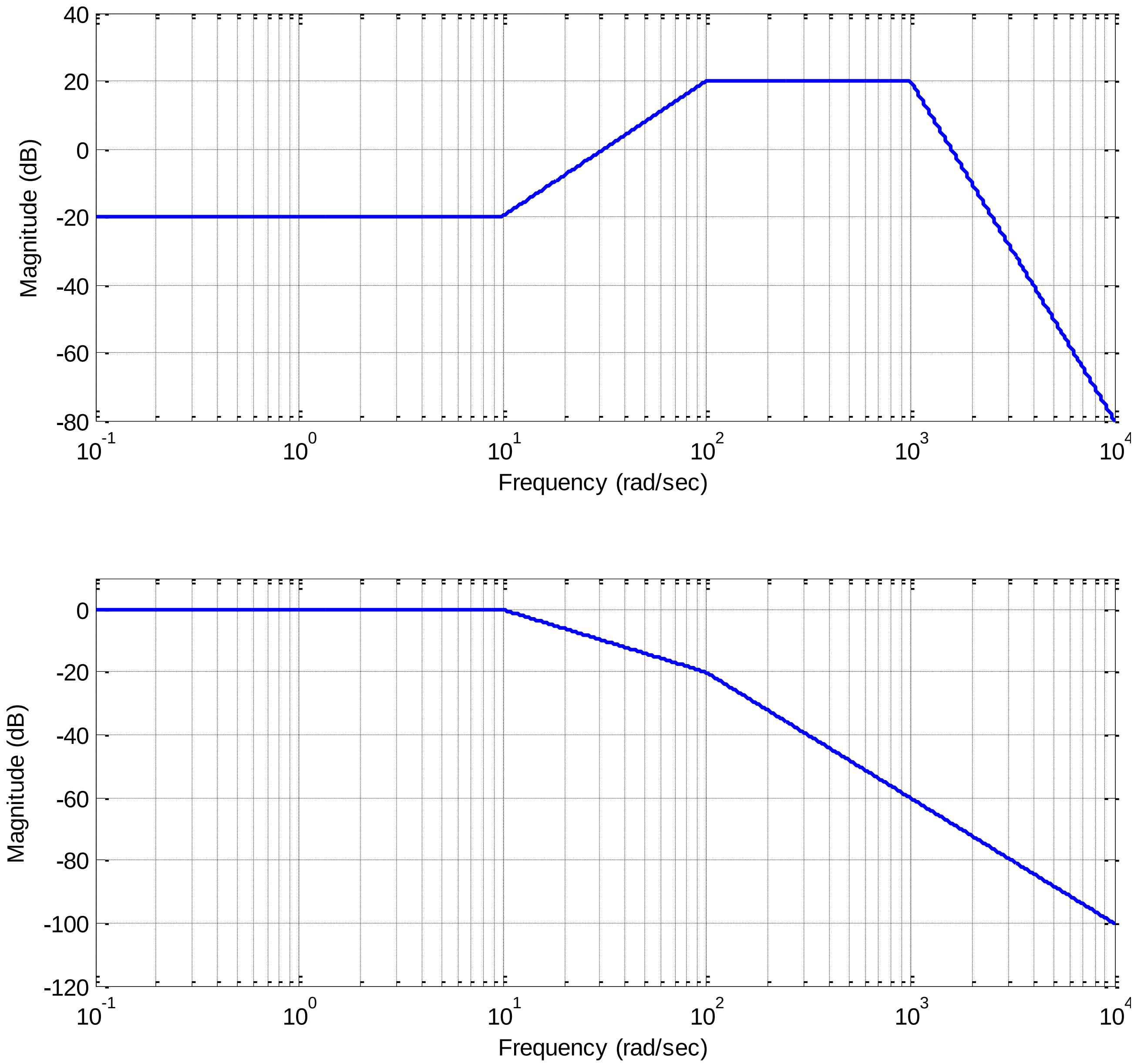


Figure 3: Problem 9, Systems e and f

Answers:

$$H(s)=\frac{100}{s^2},\quad H(s)=\frac{0.1\left(\frac{1}{100}s+1\right)^2}{s^2},\quad H(s)=10\left(\frac{1}{100}s+1\right)$$

$$H(s)=\frac{0.1\left(\frac{1}{10}s+1\right)^2}{\left(\frac{1}{1000}s+1\right)^2},\quad H(s)=\frac{0.1\left(\frac{1}{10}s+1\right)^2}{\left(\frac{1}{100}s+1\right)^2\left(\frac{1}{1000}s+1\right)^5},\quad H(s)=\frac{1}{\left(\frac{1}{10}s+1\right)\left(\frac{1}{100}s+1\right)}$$

10) Assume we want to construct a Bode plot for an LTI system.

We assume the input is $x(t)=0.087\cos(2\pi ft)$ and the corresponding steady state output is

$y_{ss}(t)=Bcos(2\pi ft+\phi)=B\cos(2\pi f(t-t_d))$

We have made the following measurements,

f	B	t_d
1.40	0.68	0.07
1.50	1.20	0.18
1.60	0.72	0.26
1.75	0.40	0.26
1.90	0.30	0.25
2.0	0.26	0.24
2.5	0.19	0.19
3.0	0.16	0.16

Plot the magnitude and phase of the transfer function using Matlab. Use a linear axis for the frequency, a logarithmic (i.e., dB) axis for the magnitude, and assume the phase is plotted in degrees.