ECE-320: Linear Control Systems
Homework 9

Due: Thursday February 13 at the beginning of class

1) Consider the continuous-time plant with transfer function

$$G_p(s) = \frac{1}{(s+1)(s+2)}$$

We want to determine the discrete-time equivalent to this plant, $G_p(z)$, by assuming a zero order hold is placed before the continuous-time plant to convert the discrete-time control signal to a continuous time control signal.

Show that if we assume a sampling interval of T, the equivalent discrete-time plant is

$$G_p(z) = \frac{z(0.5 - e^{-T} + 0.5\mathrm{e}^{-2T}) + (0.5e^{-T} - e^{-2T} + 0.5e^{-3T})}{(z-e^{-T})(z-e^{-2T})}$$

Note that we have poles were we expect them to be, but we have introduced a zero in going from the continuous time system to the discrete-time system.

2) Consider the discrete-time state variable model $\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$

where the explicit dependence of G and H on the sampling time T has been emphasized. Here

$$G(T) = e^{AT}$$
$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if A is invertible, we can write $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if A is invertible and T is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\underline{\dot{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = Ax(kT) + Bu(kT)$$

we get the same answer as in part **b,** but using this approximation we do not need to assume A is invertible.

d) Show that if we use two terms in the approximation for e^{AT} (and no assumptions about A being invertible), we can write the state equations as

$$\underline{x}(k+1) = \left[I + AT\right]\underline{x}(k) + [IT + \frac{1}{2}AT^2]Bu(k)$$

3) For the state variable system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

a) Show that

$$e^{AT} = \begin{bmatrix} 2e^{2T} - e^{3T} & e^{2T} - e^{3T} \\ 2e^{3T} - 2e^{2T} & 2e^{3T} - e^{2T} \end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

for $T = 0.1$ (integrate each entry in the matrix $e^{A\lambda}$ separately.) Compare your answer with that given by Matlab's **c2d** command, [G,H] = c2d(A,B,T).

4) For the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ show that $e^{At} = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix}$

5) Consider the discrete-time state variable model $\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$
with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k) = G_{pf}r(k) - Kx(k)$ show that the transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = C(zI - \tilde{G})^{-1}\tilde{H} = \frac{G_{pf}(z+1)}{(z+k_1)(z+k_2) - (k_1 - 1)(k_2 - 1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is the system controllable? That is, is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

6) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$ show that the transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}(z-1)}{(z-1)(z+k_2-1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is the system controllable?

6) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$ show that transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}}{z^2 + (k_2-1)z + (k_1-1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero? If we want the poles to be at p_1 and p_2 show that $k_2 = 1 - (p_1 + p_2)$ and $k_1 = 1 + p_1 p_2$.