ECE-320: Linear Control Systems
Homework 1

Due: Thursday December 5 *at the beginning of class*

Reading: Chapters 1-6, 8, 9

1) For the following transfer functions, determine both the **impulse response** and the **unit step response**.

$$H(s)=\frac{s}{(s+1)(s+2)^2} \quad H(s)=\frac{1}{(2s+1)(3s+2)}$$
$$H(s)=\frac{2}{s^2+8s+25} \quad H(s)=\frac{s+2}{s^2+2s+4}$$

Scrambled Answers:

$$h(t)=\frac{2}{3}e^{-4t}\sin(3t)u(t),\ h(t)=-e^{-t}u(t)+e^{-2t}u(t)+2te^{-2t}u(t),\ h(t)=e^{-t/2}u(t)-e^{-2t/3}u(t),$$
$$h(t)=e^{-t}cos(\sqrt{3}t)u(t)+\frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)u(t),\ y(t)=\frac{1}{2}u(t)-2e^{-t/2}u(t)+\frac{3}{2}e^{-2t/3}u(t),$$
$$y(t)=\frac{1}{2}u(t)+\frac{1}{2\sqrt{3}}e^{-t}\sin(\sqrt{3}t)u(t)-\frac{1}{2}e^{-t}\cos(\sqrt{3}t)u(t),\ y(t)=e^{-t}u(t)-e^{-2t}u(t)-te^{-2t}u(t),$$
$$y(t)=\frac{2}{25}u(t)-\frac{8}{75}e^{-4t}\sin(3t)u(t)-\frac{2}{25}e^{-4t}cos(3t)u(t)$$

2) For the following transfer functions

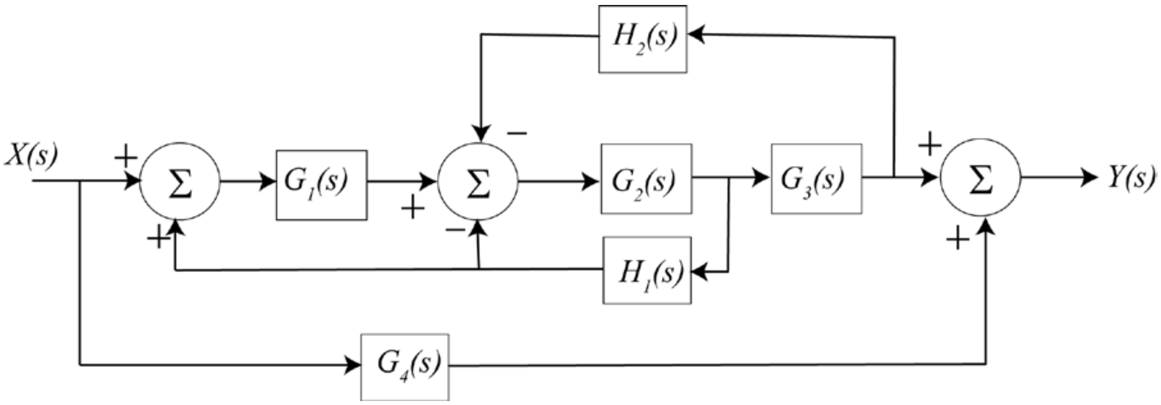
$$H(s)=\frac{2}{s^2+2s+2} \quad H(s)=\frac{3}{s^2+4s+6} \quad H(s)=\frac{5}{s^2+6s+10}$$
$$H(s)=\frac{4}{s^2-4s+7} \quad H(s)=\frac{1}{s^2+4}$$

By computing the inverse Laplace transform show that the step responses are given by

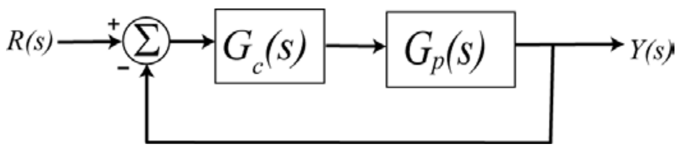
$$y(t)=\left[1-e^{-t}\cos(t)-e^{-t}\sin(t)\right]u(t) \quad y(t)=\left[\frac{1}{2}-\frac{1}{\sqrt{2}}e^{-2t}\sin(\sqrt{2}t)-\frac{1}{2}e^{-2t}\cos(\sqrt{2}t)\right]u(t)$$
$$y(t)=\left[\frac{1}{2}-\frac{3}{2}e^{-3t}\sin(t)-\frac{1}{2}e^{-3t}\cos(t)\right]u(t) \quad y(t)=\left[\frac{4}{7}+\frac{8\sqrt{3}}{21}e^{2t}\sin(\sqrt{3}t)-\frac{4}{7}e^{2t}\cos(\sqrt{3}t)\right]u(t)$$
$$y(t)=\left[\frac{1}{4}-\frac{1}{4}\cos(2t)\right]u(t)$$

3) (Mason's Rule) For the block diagram shown below, determine a corresponding signal flow diagram and show that the closed loop transfer function is

$$H_{system} = \frac{G_1G_2G_3 + G_4(1 - G_1G_2H_1 + G_2H_1 + G_2G_3H_2)}{1 - G_1G_2H_1 + G_2H_1 + G_2G_3H_2}$$



4) (Model Matching) Consider the following closed loop system, with plant $G_p(s)$ and controller $G_c(s)$.



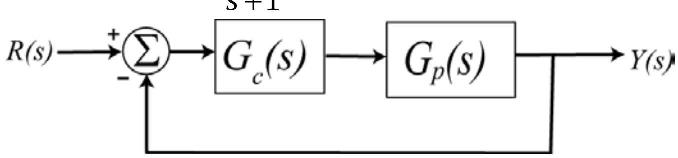
One way to choose the controller is to try and make your closed loop system match a transfer function that you choose (hence the name model matching). Let's assume that our **desired** closed loop transfer function, $G_o(s)$, our plant can be written in terms of numerators and denominators as

$$G_o(s) = \frac{N_o(s)}{D_o(s)} \quad G_p(s) = \frac{N_p(s)}{D_p(s)}$$

Show that our controller is then $G_c(s) = \dfrac{N_o(s)D_p(s)}{N_p(s)\left[D_o(s) - N_o(s)\right]}$

Note that there are some restrictions here, in that for implementation purposes the controller must be stable, and it must be proper.

5) For the following system, with plant $G_p(s) = \dfrac{1}{s+1}$, and controller $G_c(s)$



a) Using the results from problem 4, determine the controller so that the closed loop system matches a second order ITAE (**I**ntegral of **T**ime and **A**bsolute **E**rror) optimal system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

Anwes. $G_c(s) = \dfrac{\omega_0^2(s+1)}{s(s+1.4\omega_0)}$ *, note that there is a pole/zero cancellation between the controller and the plant and there is a pole at zero in the controller.*

b) Show that the damping ratio for this system is 0.7, the closed loop poles of this system are at $-0.7\omega_0 \pm j0.714\omega_0$. For faster response should ω_0 be large or small?

c) Determine the controller so that the closed loop system matches a third order **deadbeat** system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^3}{s^3 + 1.90\omega_0 s^2 + 2.20\omega_0^2 s + \omega_0^3}$$

Ans. $G_c(s) = \frac{\omega_0^3(s+1)}{s(s^2 + 1.9\omega_0 s + 2.20\omega_0^2)}$ *, note that there is a pole/zero cancellation between the controller and the plant and there is a pole at zero in the controller.*

6) One of the methods that can be used to identify ζ and ω_n for mechanical systems the *log-decrement* method, which we will derive in this problem. If our system is at rest and we provide the mass with an initial displacement away from equilibrium, the response due to this displacement can be written

$$x_1(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

where

$x_1(t)$ = displacement of the mass as a function of time

ζ = damping ratio

ω_n = natural frequency

ω_d = damped frequency = $\omega_n\sqrt{1-\zeta^2}$

After the mass is released, the mass will oscillate back and forth with period given by $T_d = \frac{2\pi}{\omega_d}$, so if we measure the period of the oscillation (T_d) we can estimate ω_d .Let's assume t_0 is the time of one peak of the cosine. Since the cosine is periodic, subsequent peaks will occur at times given by $t_n = t_0 + nT_d$, where n is an integer.

a) Show that

$$\frac{x_1(t_0)}{x_1(t_n)} = e^{\zeta\omega_n T_d n}$$

b) If we define the log decrement as

$$\delta = \ln\left[\frac{x_1(t_0)}{x_1(t_n)}\right]$$

show that we can compute the damping ratio as

$$\zeta = \frac{\delta}{\sqrt{4n^2\pi^2 + \delta^2}}$$

c) Given the initial condition response shown in the Figures on the next page, estimate the damping ratio and natural frequency using the log-decrement method. (*You should get answers that include the numbers 15, 0.2, 0.1 and 15, approximately.*)

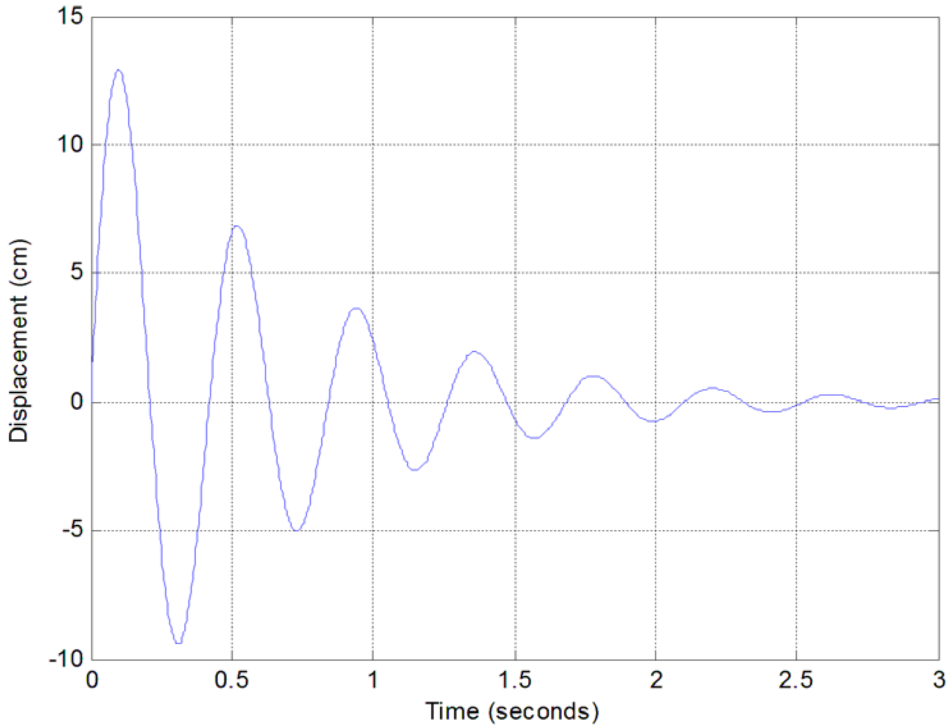


Figure 1. Initial condition response for second order system A.

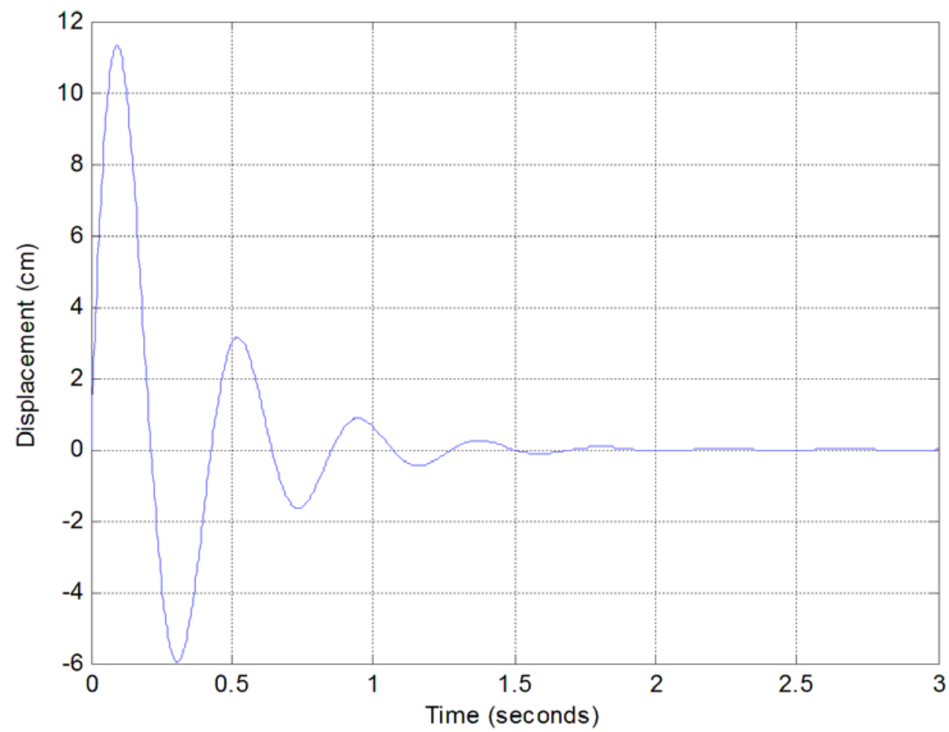


Figure 2. Initial condition response for second order system B.