ECE-320: Linear Control Systems
Homework 4

Due: **Thursday** January 10

1) Consider the plant

$$G_p(s) = \frac{\alpha_0}{s+\alpha_1} = \frac{3}{s+0.5}$$

where 3 is the nominal value of α_0 and 0.5 is the nominal value of α_1. In this problem we will investigate the sensitivity of closed loop systems with various types of controllers to these two parameters. We will assume we want the settling time of our system to be 0.5 seconds and the steady state error for a unit step input to be less than 0.1.

a) (*ITAE Model Matching*) Since this is a first order system, we will use the first order ITAE model,

$$G_o(s) = \frac{\omega_o}{s+\omega_o}$$

i) For what value of ω_o will we meet the settling time requirements and the steady state error requirements?

ii) Determine the corresponding controller $G_c(s)$.

iii) Show that the closed loop transfer function (using the parameterized form of $G_p(s)$ and the controller from part ii) is

$$G_o(s) = \frac{\frac{8}{3}\alpha_0(s+0.5)}{s(s+\alpha_1)+\frac{8}{3}\alpha_0(s+0.5)}$$

iv) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s}{s+8}$$

v) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2+8.5s+4}$$

b) (*Proportional Control*) Consider a proportional controller, with $k_p = 2.5$.

i) Show that the closed loop transfer function is

$$G_o(s) = \frac{2.5\alpha_0}{s + \alpha_1 + 2.5\alpha_0}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s + 0.5}{s + 8}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5}{s + 8}$$

c) (*Proportional+Integral Control*) Consider a PI controller with $k_p = 4$ and $k_i = 40$.

i) Show that the closed loop transfer function is

$$G_o(s) = \frac{4\alpha_0(s+10)}{s(s+\alpha_1) + 4\alpha_0(s+10)}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s(s+0.5)}{s^2 + 12.5s + 120}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2 + 12.5s + 120}$$

d) Using Matlab, simulate the unit step response of each type of controller. Plot all responses on one graph. Use different line types and a legend. Turn in your plot and code.

e) Using Matlab and subplot, plot the sensitivity to α_0 for each type of controller on ***one graph*** at the top of the page, and the sensitivity to α_1 on one graph on the bottom of the page. Be sure to use different line types and a legend. Turn in your plot and code. Only plot up to about 8 Hz (50 rad/sec) using a semilog scale with the sensitivity in dB (see below). ***Do not*** *make separate graphs for each system!*

In particular, these results should show you that the model matching method, which essentially tries and cancel the plant, are generally more sensitive to getting the plant parameters correct than the PI controller for low frequencies. However, for higher frequencies the methods are all about the same.

Hint: If $T(s) = \dfrac{2s}{s^2 + 2s + 10}$, *plot the magnitude of the frequency response using:*

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T = tf([2 0],[1 2 10]);
w = logspace(-1,1.7,1000);
[M,P]= bode(T,w);
Mdb = 20*log10(M(:));
semilogx(w,Mdb); grid;
xlabel('Frequency (rad/sec)');
ylabel('dB');
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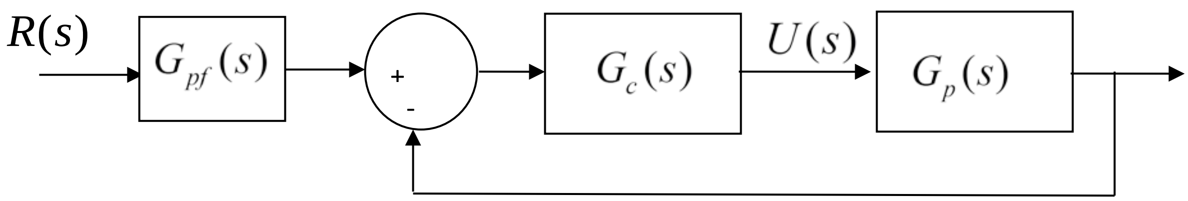
2) One of the things that will be coming up in lab more and more is the limitation of the amplitude of the control signal, or the *control effort*. This is also a problem for most practical systems. In this problem we will do some simple analysis to better understand why Matlab's sisotool won't give us a good estimate of the control effort for some types of systems, and why dynamic prefilters can often really help us out here.

a) For the system below,



show that U(s) and R(s) are related by

$$U(s) = \frac{G_c(s)G_{pf}(s)}{1+G_p(s)G_c(s)} R(s)$$

b) For many types of controllers, the maximum value of the control signal is just after the step is applied, at $t = 0^+$. Although most of the time we are concerned with steady state values and use the final value Theorem in the *s*-plane, in this case we want to use the initial value Theorem, which can be written as

$$\lim_{t \to 0^+} u(t) = \lim_{s \to \infty} sU(s)$$

If the system input is a step of amplitude *A*, show that

$$u(0^+) = \lim_{s \to \infty} \frac{AG_c(s)G_{pf}(s)}{1+G_p(s)G_c(s)}$$

This result shows very clearly that the initial control signal is directly proportional to the amplitude of the input signal, which is pretty intuitive.

c) Now let's assume

$$G_c(s) = \frac{N_c(s)}{D_c(s)} \quad G_p(s) = \frac{N_p(s)}{D_p(s)} \quad G_{pf}(s) = \frac{N_{pf}(s)}{D_{pf}(s)}$$

If we want to look at the initial value for a unit step, we need to look at

$$u(0^+) = \lim_{s \to \infty} \frac{sG_c(s)G_{pf}(s)}{1+G_c(s)G_p(s)} \frac{1}{s} = \lim_{s \to \infty} \frac{G_c(s)G_{pf}(s)}{1+G_c(s)G_p(s)}$$

Let's also then define

$$\tilde{U}(s) = \frac{G_c(s)G_{pf}(s)}{1 + G_c(s)G_p(s)}$$

so that

$$u(0^+) = \lim_{s\to\infty} \tilde{U}(s)$$

Show that

$$\tilde{U}(s) = \frac{N_{pf}(s)}{\left(\frac{D_c(s)}{N_c(s)}\right)D_{pf}(s) + \left(\frac{N_p(s)}{D_p(s)}\right)D_{pf}(s)}$$

and

$$\deg \tilde{U} = \deg N_{pf} - \max\left[\deg D_c - \deg N_c + \deg D_{pf}, \deg N_p - \deg D_p + \deg D_{pf}\right]$$

where $\deg Y$ is the degree of polynomial *Y*.

d) Since we are going to take the limit as $s \to \infty$, we need the degree of $\tilde{U}(s)$ to be less than or equal to zero for a step input to have a finite $u(0^+)$. Why?

For our 1 dof systems in lab, we have $\deg N_p = 0$ *and* $\deg D_p = 2$. *Use this for the remainder of this problem*

e) If the prefilter is a constant, show that in order to have a finite $u(0^+)$ we must have

$$\deg D_c \geq \deg N_c$$

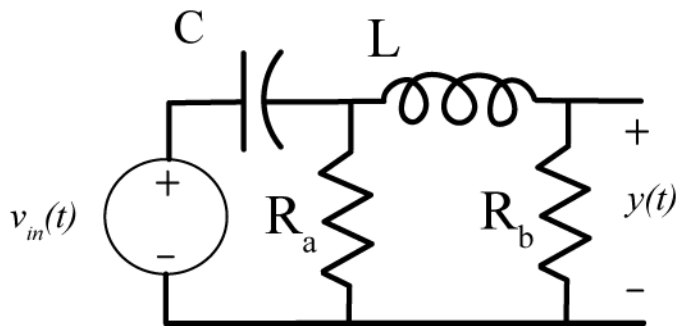
f) If the numerator of the prefilter is a constant, then in order to have a finite $u(0^+)$ we must have

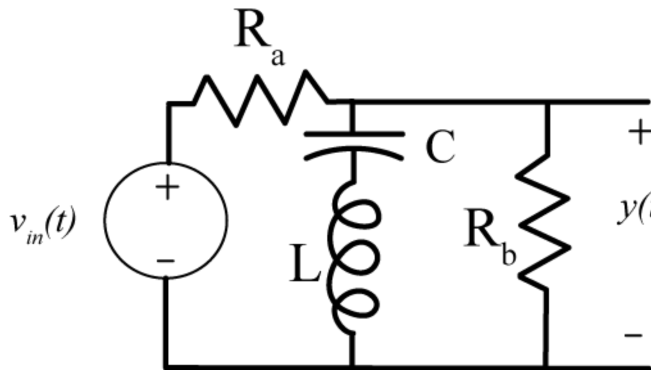
$$\deg D_c - \deg N_c + \deg D_{pf} \geq 0 \text{ or } -2 + \deg D_{pf} \geq 0$$

g) For P, I, D, PI, PD, PID, and lead controllers, determine if $u(0^+)$ is finite if the prefilter is a constant.

Note: Although it may appear that the control effort is sometimes infinite, in practice this is not true since our motor cannot produce an infinite signal. This large initial control signal is referred to as a *set-point kick*. There are different ways to implement a PID controller to avoid this, and we will cover two of them in Lab 4.

3) For the following two circuits,





show that the state variable descriptions are given by

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_b}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_a C} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_a C} \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} R_B & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + [0] v_{in}(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_a R_b}{L(R_a + R_b)} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{L(R_a + R_b)} \\ 0 \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} -\frac{R_a R_b}{R_a + R_b} & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{R_a + R_b} \end{bmatrix} v_{in}(t)$$