ECE-320: Linear Control Systems
Homework 4

Due: Tuesday January 10 at the beginning of class

1) Consider the plant

$$G_p(s) = \frac{\alpha_0}{s+\alpha_1} = \frac{3}{s+0.5}$$

where 3 is the nominal value of α_0 and 0.5 is the nominal value of α_1. In this problem we will investigate the sensitivity of closed loop systems with various types of controllers to these two parameters. We will assume we want the settling time of our system to be 0.5 seconds and the steady state error for a unit step input to be less than 0.1.

a) (*ITAE Model Matching*) Since this is a first order system, we will use the first order ITAE model,

$$G_o(s) = \frac{\omega_o}{s+\omega_o}$$

i) For what value of ω_o will we meet the settling time requirements and the steady state error requirements?

ii) Determine the corresponding controller $G_c(s)$.

iii) Show that the closed loop transfer function (using the parameterized form of $G_p(s)$ and the controller from part ii) is

$$G_o(s) = \frac{\frac{8}{3}\alpha_0(s+0.5)}{s(s+\alpha_1)+\frac{8}{3}\alpha_0(s+0.5)}$$

iv) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s}{s+8}$$

v) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2+8.5s+4}$$

b) (*Proportional Control*) Consider a proportional controller, with $k_p = 2.5$.

i) Show that the closed loop transfer function is

$$G_o(s) = \frac{2.5\alpha_0}{s+\alpha_1+2.5\alpha_0}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s+0.5}{s+8}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5}{s+8}$$

c) (*Proportional+Integral Control*) Consider a PI controller with $k_p = 4$ and $k_i = 40$.

i) Show that the closed loop transfer function is

$$G_o(s) = \frac{4\alpha_0(s+10)}{s(s+\alpha_1)+4\alpha_0(s+10)}$$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by

$$S_{\alpha_0}^{G_0} = \frac{s(s+0.5)}{s^2+12.5s+120}$$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by

$$S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2+12.5s+120}$$

d) Using Matlab, simulate the unit step response of each type of controller. Plot all responses on one graph. Use different line types and a legend. Turn in your plot and code.

g) Using Matlab and subplot, plot the sensitivity to α_0 for each type of controller on ***one graph*** at the top of the page, and the sensitivity to α_1 on one graph on the bottom of the page. Be sure to use different line types and a legend. Turn in your plot and code. Only plot up to about 8 Hz (50 rad/sec) using a semilog scale with the sensitivity in dB (see below). ***Do not*** *make separate graphs for each system!*

In particular, these results should show you that the model matching method, which essentially tries and cancel the plant, are generally more sensitive to getting the plant parameters correct than the PI controller for low frequencies. However, for higher frequencies the methods are all about the same.

Hint: If $T(s) = \frac{2s}{s^2+2s+10}$, *plot the magnitude of the frequency response using:*

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T = tf([2 0],[1 2 10]);
w = logspace(-1,1.7,1000);
[M,P]= bode(T,w);
Mdb = 20*log10(M(:));
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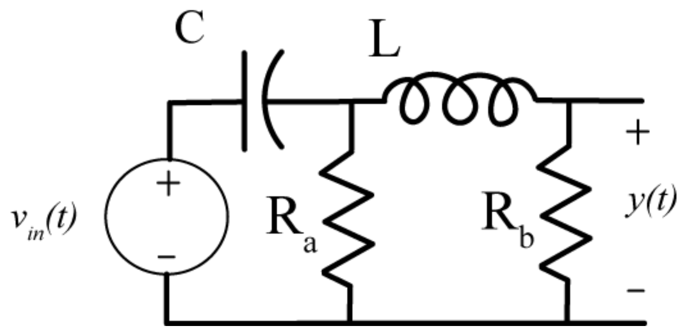
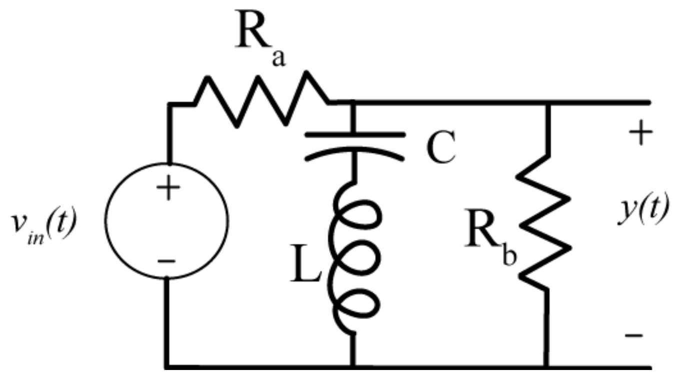
semilogx(w,Mdb); grid;
xlabel('Frequency (rad/sec)');
ylabel('dB');

2) For the following two circuits,





show that the state variable descriptions are given by

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_b}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_a C} \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_a C} \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} R_B & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + [0] v_{in}(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_a R_b}{L(R_a + R_b)} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_a}{L(R_a + R_b)} \\ 0 \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} -\frac{R_a R_b}{R_a + R_b} & 0 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_a}{R_a + R_b} \end{bmatrix} v_{in}(t)$$

3) For the plant

$$G_p(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

a) If the plant input is $u(t)$ and the output is $x(t)$, show that we can represent this system with the differential equation

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = K\omega_n^2 u(t)$$

b) Assuming we use states $q_1(t) = x(t)$ and $q_2(t) = \dot{x}(t)$, and the output is $x(t)$, show that we can write the state variable description of the system as

$$\frac{d}{dt}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

or

$$\dot{q}(t) = Aq(t) + Bu(t) \quad y(t) = Cq(t) + Du(t)$$

Determine the A, B, C and D matrices.

c) Assume we use state variable feedback of the form $u(t) = G_{pf}r(t) - kq(t)$, where $r(t)$ is the new input to the system, G_{pf} is a prefilter (for controlling the steady state error), and k is the state variable feedback gain vector. Show that the state variable model for the closed loop system is

$$\dot{q}(t) = (A - Bk)q(t) + (BG_{pf})r(t)$$
$$y(t) = (C - Dk)q(t) + (DG_{pf})r(t)$$

or

$$\dot{q}(t) = \tilde{A}q(t) + \tilde{B}r(t)$$
$$y(t) = \tilde{C}q(t) + \tilde{D}r(t)$$

d) Show that the transfer function (matrix) for the closed loop system between input and output is given by

$$G(s) = \frac{Y(s)}{R(s)} = (C - Dk)(sI - (A - Bk))^{-1}BG_{pf} + DG_{pf}$$

and if D is zero this simplifies to

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - (A - Bk))^{-1}BG_{pf}$$

e) Assume $r(t) = u(t)$ and $D = 0$. Show that, in order for $\lim_{t\to\infty} y(t) = 1$, we must have

$$G_{pf} = \frac{-1}{C(A - Bk)^{-1}B}$$

Note that the prefilter gain is a function of the state variable feedback gain!

If matrix P is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the determinant of P is given by $ad-bc$. This determinant will also give us the characteristic polynomial of the system.

4) For each of the systems below:

- determine the transfer function when there is state variable feedback
- determine if k_1 and k_2 exist ($k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$) to allow us to place the closed loop poles anywhere. That is, can we make the denominator look like $s^2 + a_1 s + a_0$ for any a_1 and any a_0. If this is true, the system is said to be ***controllable***.

a) Show that for

$$\dot{q} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{(s-1)G_{pf}}{(s-1)(s-1+k_2)}$

b) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{sG_{pf}}{s^2 + (k_2-1)s + k_1}$

c) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{G_{pf}}{s^2 + (k_2-1)s + (k_1-1)}$