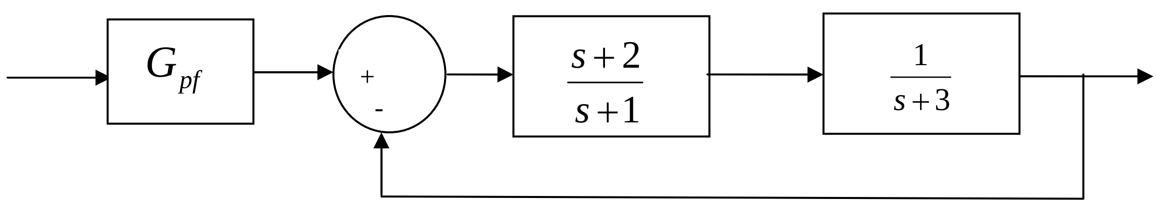
ECE-320: Linear Control Systems
Homework 3

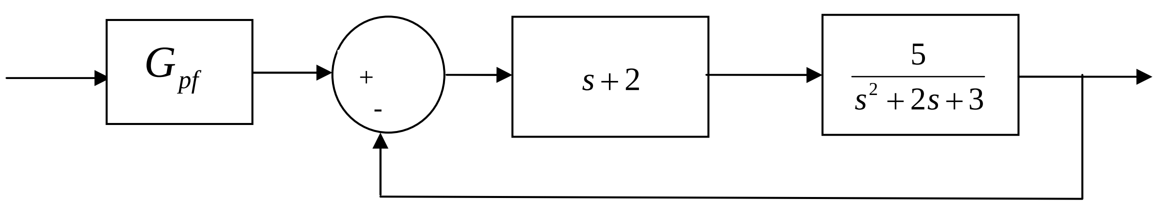
Due: Thursday December 17 at the beginning of class

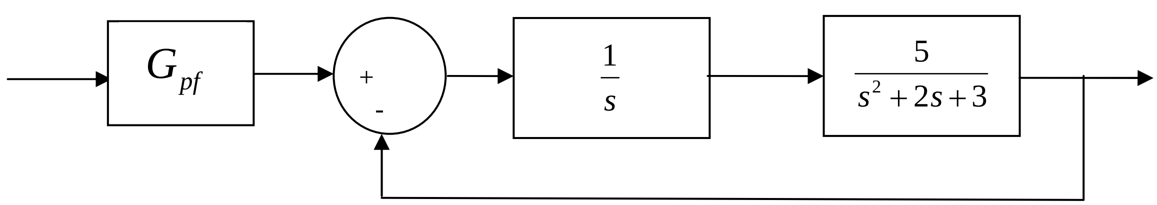
1) For the following systems

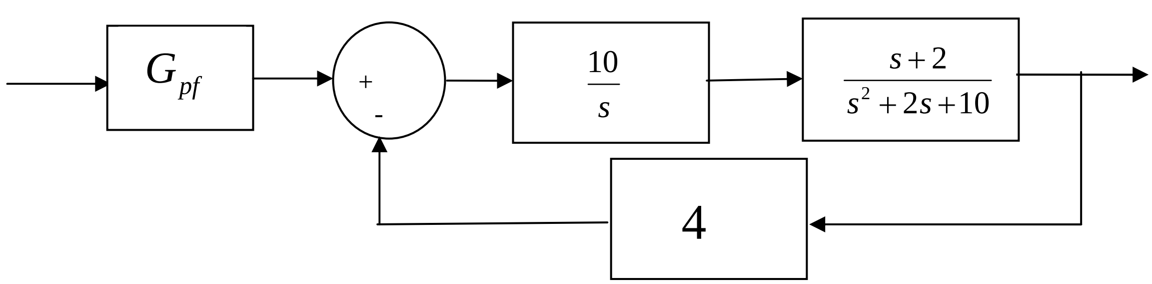
a) Determine the value of the prefilter gain G_{pf} so that the steady state error for a unit step input of the closed loop system is zero. *(Hint: Think about the easiest way to do this, you should never have to write out the closed loop transfer function as the ratio of two polynomials.) Ans. 1,4,1.3,2.5 (though not in that order)*

b) Simulate each system in Matlab (not Simulink) for a step response. Run the simulation until the system comes to steady state. Use **subplot** to put all four plots on one page. You may want to type **orient tall** before your first plot so Matlab will use more of the page.

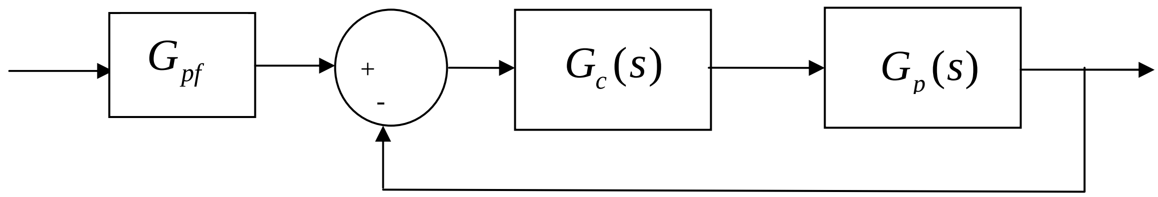








2) For the following system, with prefilter G_{pf}, plant $G_p(s) = \dfrac{1}{s+1}$, and controller $G_c(s)$



a) Assume the prefilter is absent (i.e., it's a 1), and determine the controller so that the closed loop system matches a second order ITAE optimal system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

Ans. $G_c(s) = \dfrac{\omega_0^2(s+1)}{s(s+1.4\omega_0)}$, *note that there is a pole/zero cancellation between the controller and the plant and there is a pole at zero in the controller.*

b) Show that the damping ratio for this system is 0.7, the closed loop poles of this system are at $-0.7\omega_0 \pm j0.714\omega_0$. For faster response should ω_0 be large or small?

c) If we want the steady state error for a step input to be zero, what should the value of the prefilter be?

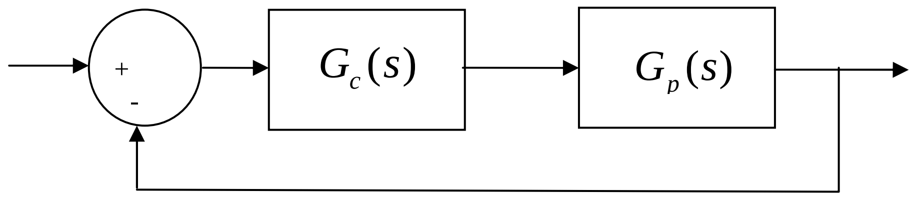
d) Assume the prefilter is absent (i.e., it's a 1) and determine the controller so that the closed loop system matches a third order deadbeat system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^3}{s^3 + 1.90\omega_0 s^2 + 2.20\omega_0^2 s + \omega_0^3}$$

Ans. $G_c(s) = \dfrac{\omega_0^3(s+1)}{s(s^2 + 1.9\omega_0 s + 2.20\omega_0^2)}$, *note that there is a pole/zero cancellation between the controller and the plant and there is a pole at zero in the controller.*

e) If we want the steady state error for a step input to be zero, what should the value of the prefilter be?

3) Assume we have the following system



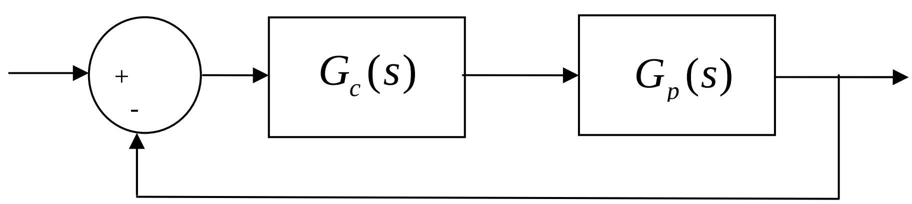
where $G_p(s) = \dfrac{1}{s+2}$. We want to design a model matching controller so that

- the closed loop system is a second order
- the state error for a step input is zero
- the settling time of the closed loop system is 3 seconds
- the percent overshoot of the closed loop system is 20%

. You are to determine the closed loop transfer function and then the controller to meet these specifications.

Ans. $G_c(s) = \dfrac{8.53(s+2)}{s(s+2.66)}$

4) Assume we have the following system



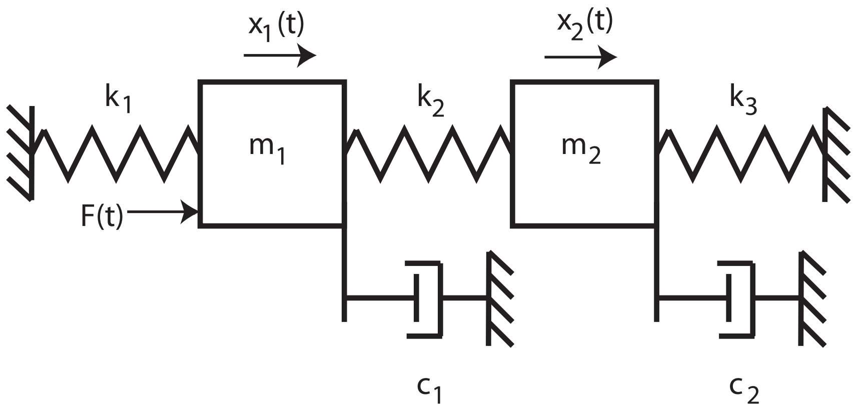
where $G_p(s) = \dfrac{1}{s+2}$. We want to design a model matching controller so that

- the closed loop system is a second order
- the steady state error for a step input is zero
- the bandwidth of the closed loop system is 2 Hz.

Luckily I have an uncle who sells good poles cheap, so I obtained a pole at -10π rad/sec for your system. You are to determine the closed loop transfer function (i.e., find the other pole and the numerator) and then the controller to meet these specifications. (Hint: *Read Chapter 7 of the notes!*, *Ans.* $G_c(s) = \dfrac{40\pi^2(s+2)}{s(s+14\pi)}$)

Preparation for Lab 3

5) Consider the following model of the two degree of freedom system we will be using in lab 3.



a) Draw free body diagrams for each mass and show that the equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 &= F + k_2 x_2 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 &= k_2 x_1 \end{aligned}$$

If we define $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, and $q_4 = \dot{x}_2$, then we get the following *state equations*

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & -\left(\frac{c_1}{m_1}\right) & \left(\frac{k_2}{m_1}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{k_2}{m_2}\right) & 0 & -\left(\frac{k_2+k_3}{m_2}\right) & -\left(\frac{c_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{m_1}\right) \\ 0 \\ 0 \end{bmatrix} F$$

This is of the general form $\dot{q}(t) = Aq(t) + BF(t)$. In what follows the notation $A_{i,j}$ refers to the entry in the i^{th} row and j^{th} column in the above A matrix. In order to get the A and B matrices for the state variable model, we need to determine all of the quantities in the above matrices.

b) Now we will rewrite the equations from part (a) as

$$\begin{aligned} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 &= \frac{k_2}{m_1}x_2 + \frac{1}{m_1}F \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 &= \frac{k_2}{m_2}x_1 \end{aligned}$$

We will get our initial estimates of ζ_1, ω_1, ζ_2, and ω_2 using the log-decrement method (assuming only one cart is free to move at a time). Assuming we have measured these parameters, show how $A_{2,1}$, $A_{2,2}$, $A_{4,3}$, and $A_{4,4}$ can be determined.

c) By taking the Laplace transforms of the equations from part (d), show that we get the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) - \frac{k_2^2}{m_1 m_2}}$$

d) It is more convenient to write this as

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)}$$

By equating powers of s in the denominator of the transfer function from part (c) and this expression you should be able to write down four equations. The equations corresponding to the coefficients of s^3, s^2, and s do not seem to give us any new information, but they will be used to get consistent estimates of ζ_1 and ω_1. The equation for the coefficient of s^0 will give us a new relationship for $\frac{k_2^2}{m_1 m_2}$ in terms of the parameters we will be measuring.

e) We will actually be fitting the frequency response data to the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{K_2}{\left(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1\right)\left(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1\right)}$$

What is K_2 in terms of the parameters of part (d)?

f) Using the transfer function in (d) and the Laplace transform of the second equation in part (b), show that the transfer function between the input and the position of the first cart is given as

$$\frac{X_1(s)}{F(s)} = \frac{\frac{1}{m_1}\left(s^2 + 2\zeta_2\omega_2 s + \omega_2^2\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)}$$

g) This equation is more convenient to write in the form

$$\frac{X_1(s)}{F(s)} = \frac{K_1\left(\frac{1}{\omega_2^2}s^2 + \frac{2\zeta_2}{\omega_2}s + 1\right)}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)}$$

What is K_1 in terms of the quantities given in part (f)?

h) Verify that $A_{4,1} = \frac{k_2}{m_2} = \frac{K_2}{K_1}\omega_2^2$

i) Verify that $A_{2,3} = \frac{k_2}{m_1} = \frac{\omega_1^2\omega_2^2 - \omega_a^2\omega_b^2}{A_{4,1}}$

j) Verify that $B_2 = \frac{1}{m_1} = \frac{K_2\omega_a^2\omega_b^2}{A_{4,1}}$. Note that this term contains all of the scaling and unit conversions.